TOPOLOGY

In this exciting topology adventure, we generalize many of the results we've seen to $\mathbb{R}^k$ or more general spaces called *metric spaces*. Even though this is more abstract, it really gives us a nice perspective into what concepts in this course are essential, and which ones are not.

1. Metric Spaces

Video: Metric Spaces

The single, *most important* identity with absolute values that we have learned so far is the triangle inequality, which states:

Triangle Inequality

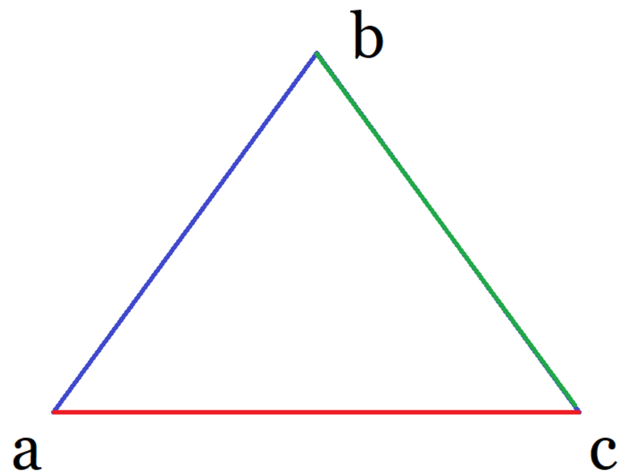
$$|x + y| \leq |x| + |y|$$

And one of its consequences was:

Corollary

$$|a - c| \leq |a - b| + |b - a|$$

Interpretation: The third leg of a triangle is shorter than the sum of the other two legs.



There are other properties of $|x|$ that we have used, even though we didn't really think of them:

For all $x, y, z \in \mathbb{R}$,

(1) $|x - y| \geq 0$

(2) $|x - y| = 0 \Leftrightarrow x = y$

(3) $|x - y| = |y - x|$

(4) $|x - z| \leq |x - y| + |y - z|$

Main Idea: What if we forget *everything* about $\mathbb{R}$ and absolute values, *except* the four properties above? Then we get an extremely useful object called a **metric space**:

Definition:

If S is any set, then[a], (S, d) is called a **metric space** if the following 4 properties hold. Here $x, y, z \in S$

(1) $d(x, y) \geq 0$

(2) $d(x, y) = 0 \Leftrightarrow x = y$

(3) $d(x, y) = d(y, x)$

(4) $d(x, z) \leq d(x, y) + d(y, z)$

[a]Here d is any function from $S \times S$ to $\mathbb{R}$

Notice the similarity between $d(x, y)$ and $|x - y|$, so in fact d (called a metric) is just a generalization of the absolute value.

To show how useful and powerful this concept is, let me give you 10 examples of metric spaces.

Note: The first 5 examples are important for your homework, but you can skip the last 5 if you wish, although they are pretty cool.

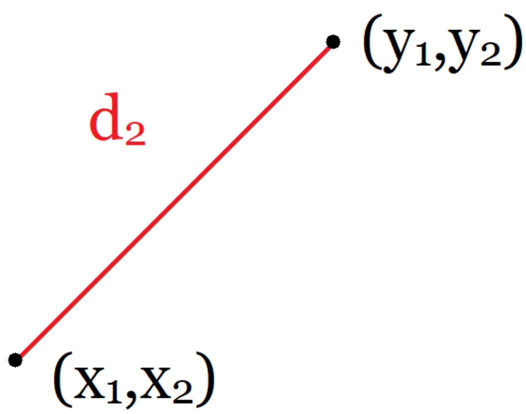
Example 1:

$(\mathbb{R}, d)$ with $d(x, y) = |x - y|$

Example 2:

$(\mathbb{R}^2, d_2)$

$$d_2((x_1, x_2), (y_1, y_2)) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$$



Example:

$$d_2((1,2),(3,4)) = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{8}$$

Note: Because this is such a natural distance function on $\mathbb{R}^2$, from now on we'll write d instead of d_2.

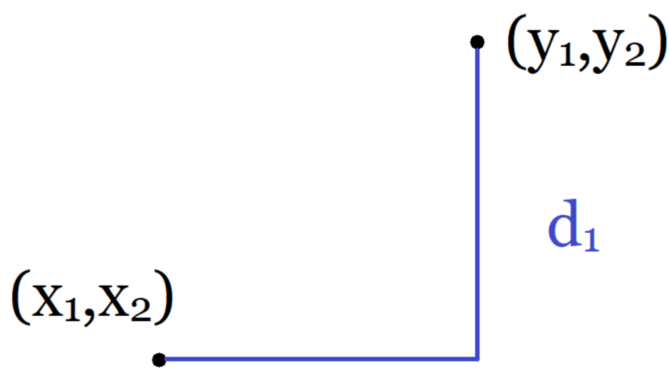
Note: This can be generalized to $\mathbb{R}^k$:

$$d_2((x_1, \dots, x_k), (y_1, \dots, y_k)) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_k - x_k)^2}$$

Example 3:

$(\mathbb{R}^2, d_1)$

$$d_1((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_2|$$



In other words, just add the sum of the lengths of the legs of the triangle

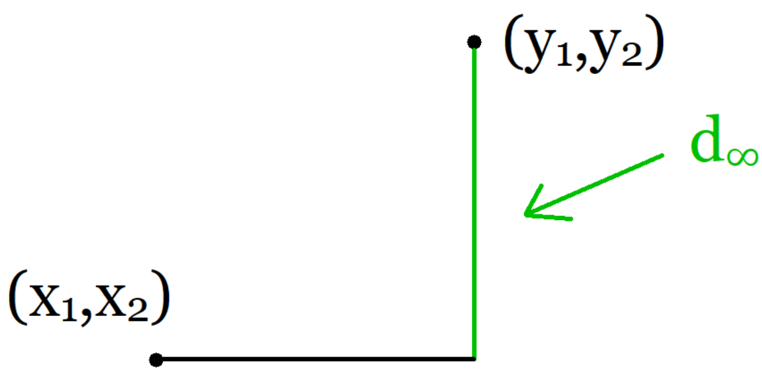
Note: This is sometimes called the taxicab (or Manhattan) metric. Because taxicabs in New York can't just go diagonally from (x_1, x_2) to (y_1, y_2) without crashing into buildings, they have to go right, and then up.

Example 4:

$(\mathbb{R}^2, d_\infty)$

$$d_\infty((x_1, x_2), (y_1, y_2)) = \max\{|y_1 - x_1| + |y_2 - x_2|\}$$

Note: In other words, just calculate the length of the *biggest* leg

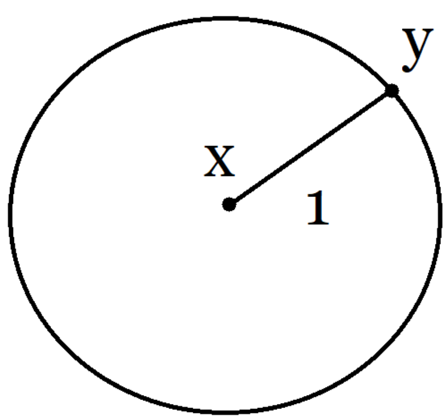


Example 5: Discrete Metric

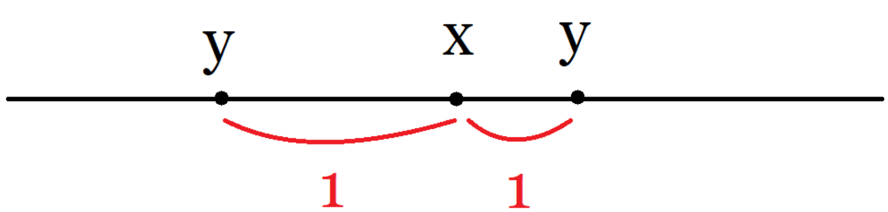
$(\mathbb{R}, d)$ with

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

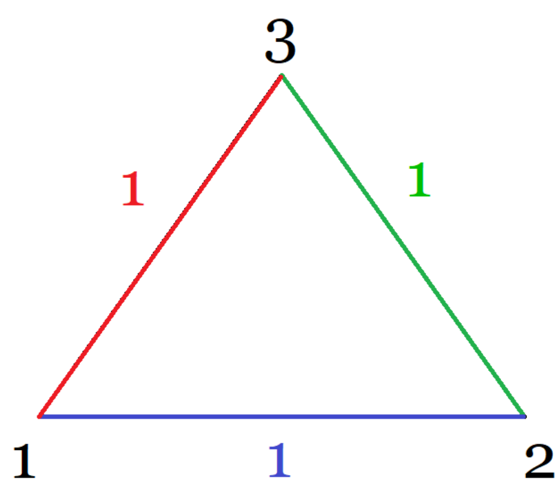
(This is Example 10 in the video)



In other words, with the metric d, *all* the points in $\mathbb{R}$ are distance 1 apart. Freaky, isn't it? But it's a great source of counterexamples!



Note: The discrete metric seems weird for $\mathbb{R}$, but is more natural in other examples:
Example: $S = \{1, 2, 3\}$ with the discrete metric. Then S is just an equilateral triangle!

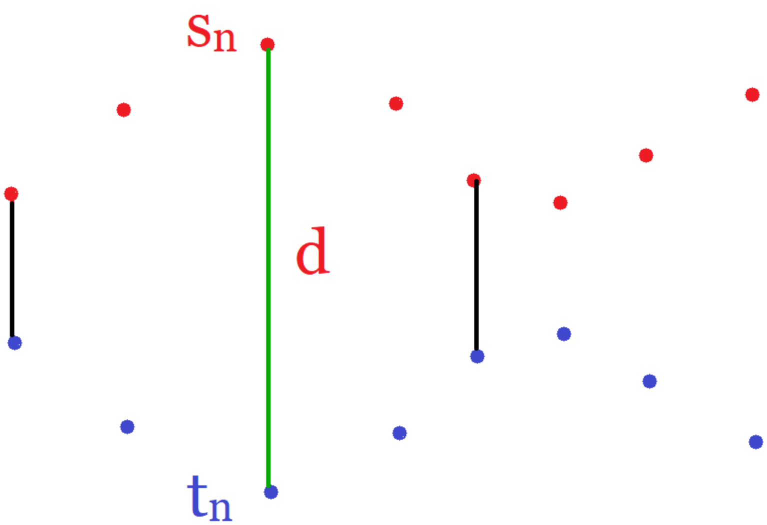


Example 6:

S = Set of bounded sequences in $\mathbb{R}$ with

$$d((s_n), (t_n)) = \sup \{|s_n - t_n| \mid n \in \mathbb{N}\}$$

In other words, look at the largest possible difference between s_n and t_n.



Note: The following is NOT a metric on S:

$$d((s_n),(t_n)) = \sum_{n=1}^{\infty} |s_n - t_n|$$

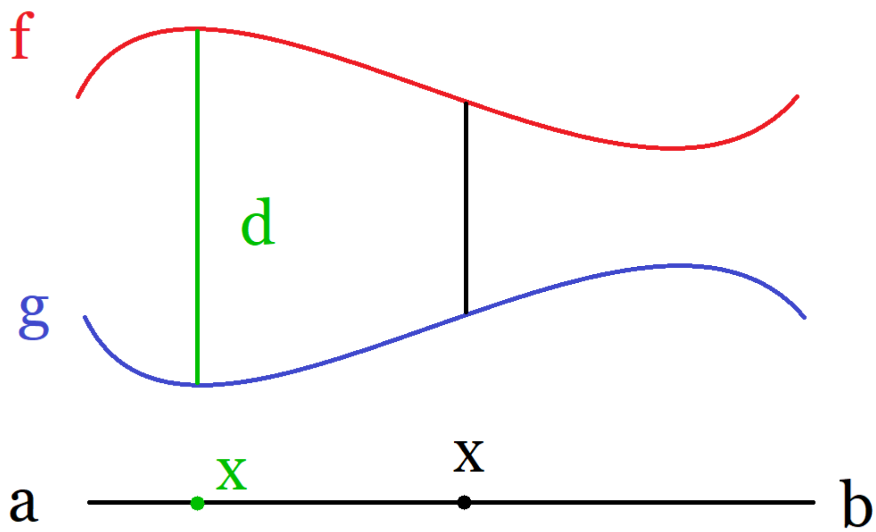
Because $d((s_n),(t_n))$ might be ∞ (for instance with $(s_n) = (1,1,1,\dots)$ and $(t_n) = (0,0,0,\dots)$), but for a metric we must have $d(x,y) < \infty$ for all x and y.

Example 7:

S = Continuous functions on $[a,b]$ (see Chapter 3) with

$$d(f,g) = \max\{|f(x) - g(x)| \mid x \in [a,b]\}$$

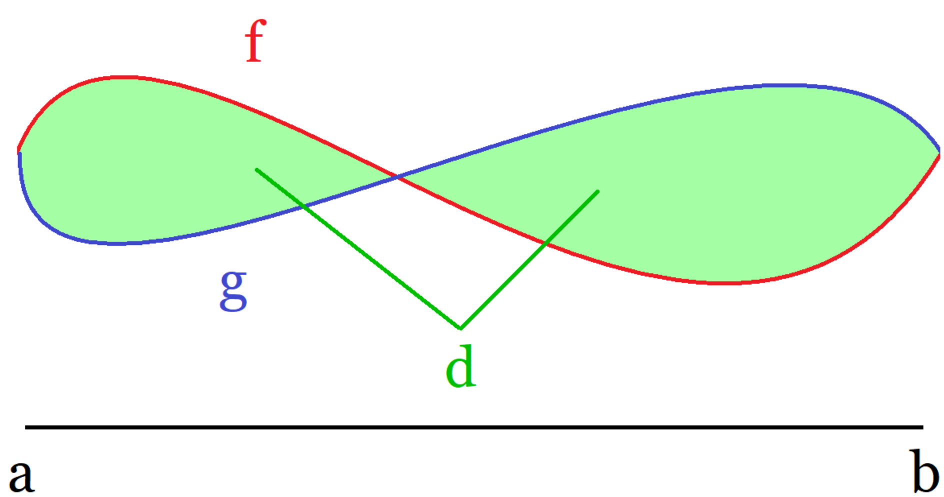
This is a continuous analog of the previous example



Example 8:

Same, but this time

$$d(f, g) = \int_a^b |f(x) - g(x)| \, dx$$



Example 9:

Same, but this time

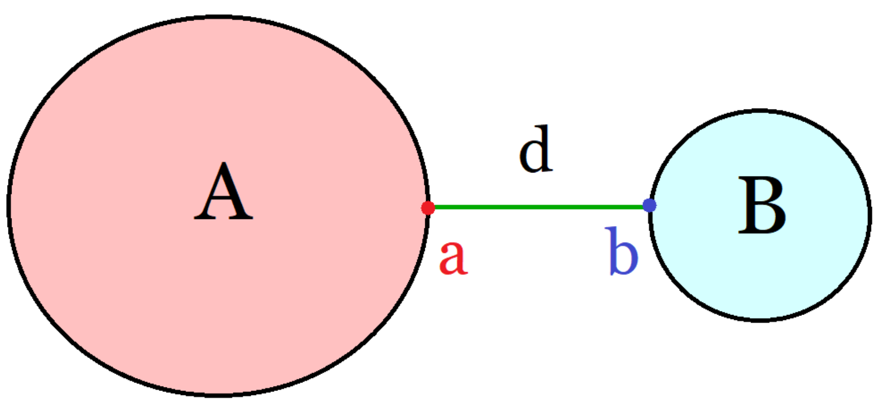
$$d(f,g) = \sqrt{\int_a^b |f(x) - g(x)|^2 \, dx}$$

This is a very natural metric on S if you remember that an integral is just a sum (so here we take the square root of the sum of squares). It's also nice because S becomes a Hilbert space (in case you know what that is)

Example 10:

If A and B are two subsets of $\mathbb{R}$ (or of any metric space), then

$$d(A,B) = \inf \{|a-b| \mid a \in A, b \in B\}$$



(Analogy: Think like two lovebirds on two different continents A and B who try to communicate with each other).

So you can even measure how far whole sets are, how cool is that?

Take-away: *Everything* we're going to show in this trilogy holds for **ALL** 10 examples at once, so we're really killing 10 birds with one stone! *THIS* is the power of abstract mathematics!

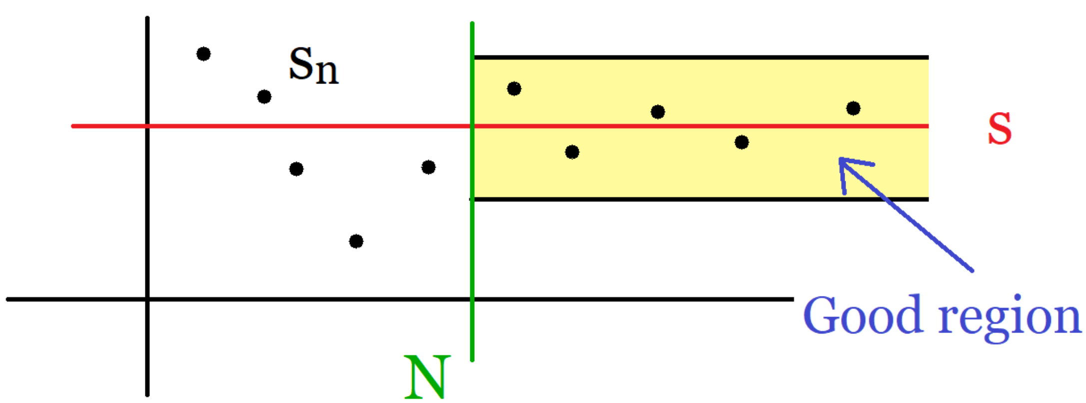
2. Convergence

Video: Convergence in $\mathbb{R}^k$

The neat thing about metric spaces is that it's really easy to generalize the notion of convergence to those spaces.

Recall:

If (s_n) is a sequence in $\mathbb{R}$, then $s_n \to s$ if for all $\epsilon > 0$ there is N such that if $n > N$, then $|s_n - s| < \epsilon$.

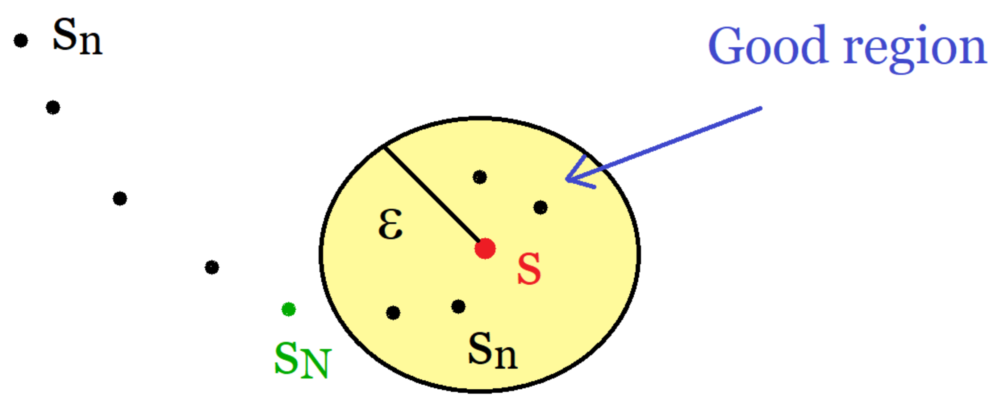


It's *exactly* the same for metric spaces, except you replace the absolute value with d !

Definition:

If (S, d) is a metric space and (s_n) is a sequence in S, then $s_n \to s$ if for all $\epsilon > 0$ there is N such that if $n > N$, then $d(s_n, s) < \epsilon$

That said, even though the definition is the same, the way of thinking is a bit different. Here the good region is a ball (Points that are at most ϵ apart from s) instead of a strip and the sequence gets closer and closer to s.



3. Convergence in $\mathbb{R}^k$

Since $\mathbb{R}^k$ *is* a metric space, in order to define convergence in $\mathbb{R}^k$, we just need to apply the definition above to the space $\mathbb{R}^k$.

Notation:

(1) Points in $\mathbb{R}^k$ will be denoted by $(x_1, \ldots, x_k)$

(2) The distance between $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ is defined as

$$d((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2}$$

(3) Sequences in $\mathbb{R}^k$ will be written as

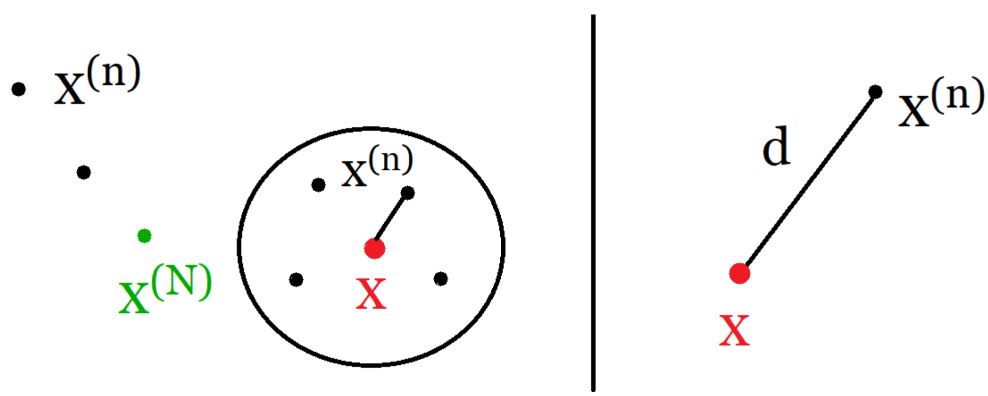
$$(x^{(n)}) = (x_1^{(n)}, \ldots, x_k^{(n)})$$

So the first term is $(x_1^{(1)}, \ldots, x_k^{(1)})$, the second term is $(x_1^{(2)}, \ldots, x_k^{(2)})$ and so forth.

With the notation as above, we can now define what it means for a sequence $(x^{(n)})$ in $\mathbb{R}^k$ to converge to x.

Definition:

If $(x^{(n)})$ is a sequence in $\mathbb{R}^k$, then we say (x^n) converges to x if for all $\epsilon > 0$ there is N such that if $n > N$, then $d(x^n, x) < \epsilon$



Luckily we never have to use this to prove that $x^{(n)}$ converges to x, because we have the following really useful result.

Motivation: Consider the following sequence in $\mathbb{R}^2$:

$$x^{(n)} = \left(\frac{1}{n}, e^{-n}\right) = \left(x_1^{(n)}, x_2^{(n)}\right)$$

Then $x^{(n)} \to (0,0) = x$.

How did we figure this out? Well, notice that $x_1^{(n)} = \frac{1}{n} \to 0$ and $x_2^{(n)} = e^{-n} \to 0$, and from this we concluded that $x^{(n)} \to x = (0,0)$.

In other words, to figure out if $x^{(n)} \to x$, it is enough to check if each component $x_1^{(n)}$ and $x_2^{(n)}$ converges to x_1 and x_2 respectively, where $x = (x_1, x_2)$. And in fact, this is always true:

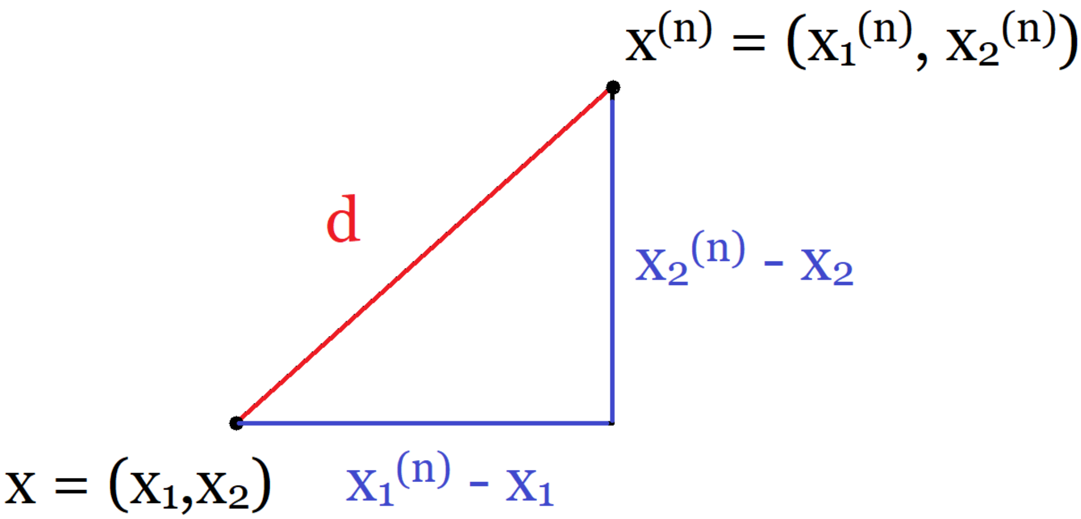
Theorem:

If $(x^{(n)}) = (x_1^{(n)}, \dots, x_k^{(n)})$ is a sequence in $\mathbb{R}^k$, then

$$(x^{(n)}) \to x \Leftrightarrow x_j^{(n)} \to x_j \text{ (for each } j = 1, \dots, k)$$

Where $x = (x_1, \dots, x_k)$

Note: In terms of triangles, this makes sense: This just says that if the legs of the triangle are small, then the hypotenuse is small. And conversely, if the hypotenuse is small, then each leg is small as well. The picture below illustrates the case $k = 2$:



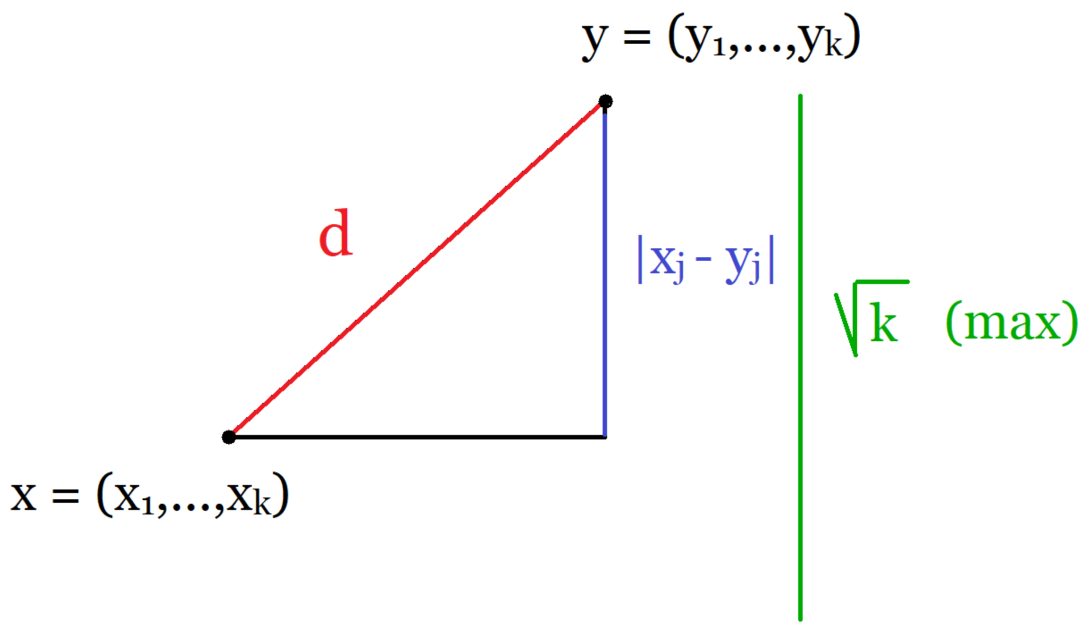
In order to prove this, we need a small lemma, which is kind of like a Squeeze Theorem, but for distances:

Lemma: [Squeeze Theorem for Distances]

If $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are points in $\mathbb{R}^k$, then for all $j = 1, \ldots, k$, we have:

$$|x_j - y_j| \leq d(x, y) \leq \sqrt{k} \max \{|x_1 - y_1|, \ldots, |x_k - y_k|\}$$

Where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$



This Lemma says two things: First of all, it says that the hypotenuse d of the triangle is bigger than each of its legs $|x_j - y_j|$. On the other hand, the hypotenuse cannot be that big either. It is always smaller than a constant (Here $\sqrt{k}$, think for instance $\sqrt{2}$ in the case of $\mathbb{R}^2$) times the biggest leg of the triangle. In the picture above, the red diagonal is smaller than the green vertical line.

Proof: On the one hand, we have:

$$\begin{aligned} d((x_1,\ldots,x_k),(y_1,\ldots,y_k)) &= \sqrt{\underbrace{(x_1-y_1)^2}_{\geq 0} + \cdots + (x_j-y_j)^2 + \cdots + \underbrace{(x_k-y_k)^2}_{\geq 0}} \\ &\geq \sqrt{(x_j-y_j)^2} \\ &= |x_j - y_j| \checkmark \end{aligned}$$

On the other hand, let $M = \max\{|x_1 - y_1|, \dots, |x_k - y_k|\}$, then each $|x_j - y_j| \leq M$

$$\begin{aligned} d((x_1, \dots, x_k), (y_1, \dots, y_k)) &= \sqrt{\underbrace{(x_1 - y_1)^2}_{\leq M^2} + \cdots + \underbrace{(x_k - y_k)^2}_{\leq M^2}} \\ &\leq \sqrt{M^2 + M^2 + \cdots + M^2} \\ &\leq \sqrt{kM^2} \\ &= \sqrt{k} M \\ &= \sqrt{k} \max\{|x_1 - y_1|, \dots, |x_k - y_k|\} \checkmark \quad \square \end{aligned}$$

Proof of Theorem: We need to show that

$$(x^{(n)}) \to x = (x_1, \dots, x_k) \Leftrightarrow x_j^{(n)} \to x_j \text{ (for each } j = 1, \dots, k)$$

($\Rightarrow$) Let $\epsilon > 0$ be given, then since $(x^{(n)}) \to x$, there is N such that if $n > N$, then $d(x^{(n)}, x) < \epsilon$. But then, for the same N, if $n > N$, then by the Lemma above, for each j, we have

$$\left| x_j^{(n)} - x_j \right| \leq d((x_1^{(n)}, \dots, x_k^{(n)}), (x_1, \dots, x_k)) = d((x^{(n)}), x) < \epsilon$$

Hence $\left| x_j^{(n)} - x_j \right| < \epsilon$ and therefore x_j^n converges to x_j ✓

($\Leftarrow$) Let $\epsilon > 0$ be given.

Then, since $x_1^{(n)}$ converges to x_1, there is N_1 such that if $n > N_1$, then $\left| x_1^{(n)} - x_1 \right| < \frac{\epsilon}{\sqrt{k}}$

Since $x_2^{(n)}$ converges to x_2, there is N_2 such that if $n > N_2$, then $\left|x_2^{(n)} - x_2\right| < \frac{\epsilon}{\sqrt{k}}$

And in general, since $x_j^{(n)}$ converges to x_j, there is N_2 such that if $n > N_j$, then $\left|x_j^{(n)} - x_j\right| < \frac{\epsilon}{\sqrt{k}}$

Now if $N = \max\{N_1, \dots, N_k\}$, then if $n > N$, by the Lemma, we have

$$\begin{aligned} d((x^{(n)}), x) =& d((x_1^{(n)}, \dots, x_k^{(n)}), (x_1, \dots, x_k)) \\ \leq& \sqrt{k} \max\left\{\underbrace{\left|x_1^{(n)} - x_1\right|}_{<\frac{\epsilon}{\sqrt{k}}}, \dots, \underbrace{\left|x_k^{(n)} - x_k\right|}_{<\frac{\epsilon}{\sqrt{k}}}\right\} \\ <& \sqrt{k}\left(\frac{\epsilon}{\sqrt{k}}\right) \\ =& \epsilon \end{aligned}$$

Hence $(x^{(n)})$ converges to x ✓ □

4. $\mathbb{R}^k$ IS COMPLETE

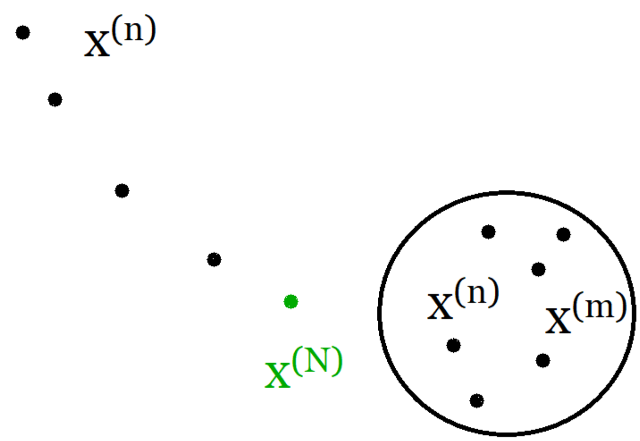
Video: $\mathbb{R}^k$ is complete

As a neat consequence of the above, we get that $\mathbb{R}^k$ is complete. To do this, let's quickly adapt the definition of Cauchy and completeness to $\mathbb{R}^k$ (the same definition is valid for metric spaces)

Definition:

If $(x^{(n)})$ is a sequence in $\mathbb{R}^k$, then we say $(x^{(n)})$ is **Cauchy** if, for all $\epsilon > 0$, there is N such that if $m, n > N$, then $d((x^{(n)}), (x^{(m)})) < \epsilon$

In other words, the terms of the sequence $x^{(n)}$ eventually get closer and closer together. Notice that this definition makes no mention of limits.



Note: Using almost the exact same proof as above, one can show that $(x^{(n)})$ is Cauchy (in $\mathbb{R}^k$) if and only if each component (x_j^n) is Cauchy (in $\mathbb{R}$)

Theorem:

$\mathbb{R}^k$ is **complete**, meaning every Cauchy sequence in $\mathbb{R}^k$ converges.

Proof: Easy! We've done the hard part above.

Let (x^n) be a Cauchy sequence in $\mathbb{R}^k$.

Then, by the above, each component $(x_j^{(n)})$ is Cauchy in $\mathbb{R}$ (for $j = 1, \dots, k$)

But since $\mathbb{R}$ is complete, each $x_j^{(n)}$ converges to some x_j.

But then by the Theorem above, $(x^{(n)})$ converges to x, where $x = (x_1, \dots, x_k)$ $\square$

5. Bolzano-Weierstrass for $\mathbb{R}^k$

Video: Bolzano-Weierstraß in $\mathbb{R}^k$

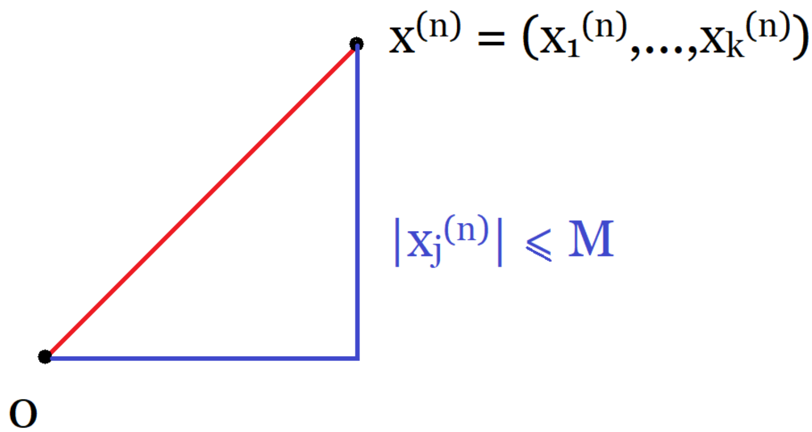
Lastly, we can generalize the Bolzano-Weierstraß Theorem to $\mathbb{R}^k$, which says that every bounded sequence has a convergent subsequence.

Bounded just means that every component is bounded:

Definition:

If $(x^{(n)})$ is a sequence in $\mathbb{R}^k$, then $(x^{(n)})$ is bounded if there is $M > 0$ such that $\left| x_j^{(n)} \right| < M$ for all n (and all $j = 1, \dots, k$)

Basically, all it means is that the length of each leg of the triangle is $\leq M$.

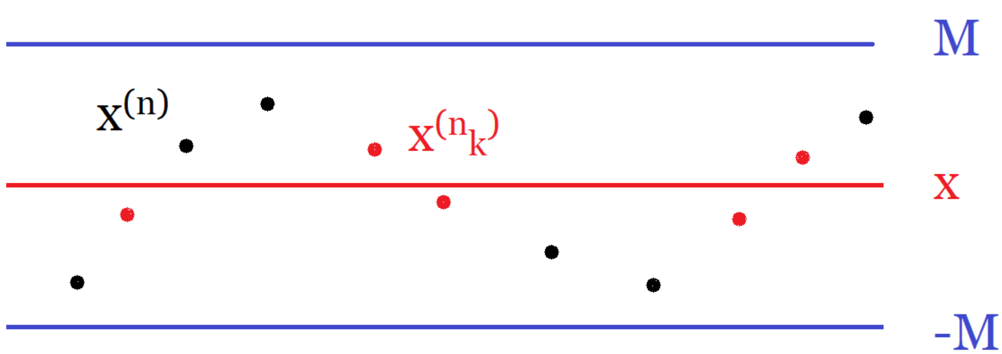


It also implies that the sequence $(x^{(n)})$ is inside a box (of side $M - (-M) = 2M$), but we won't really need that fact.

Bolzano-Weierstraß for $\mathbb{R}^k$:

Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence

(Strictly speaking, the figure above should be in $\mathbb{R}^k$)



Proof: Basically apply Bolzano-Weierstraß to each component.

STEP 1: Let $(x^{(n)})$ be a bounded sequence in $\mathbb{R}^k$

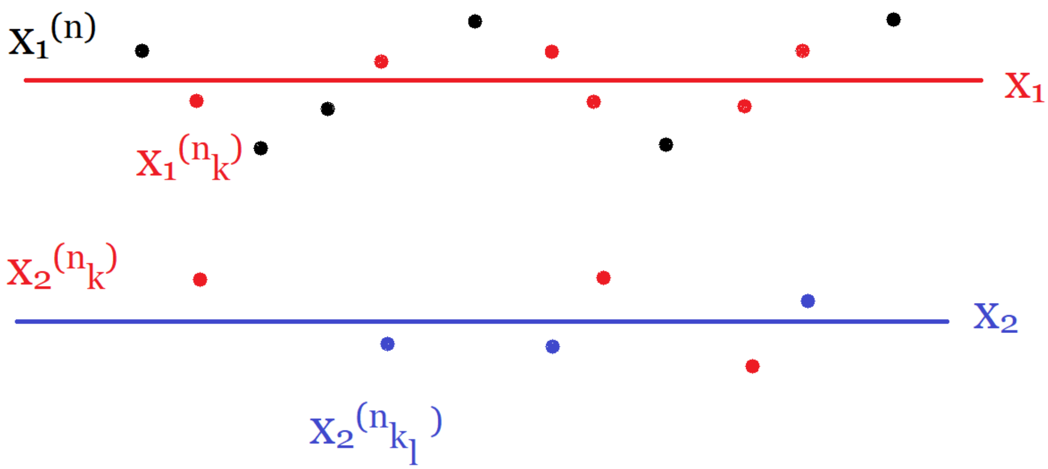
Then there is $M > 0$ such that for all $j = 1, \dots, k$, $\left|x_j^{(n)}\right| \leq M$.

But with $j = 1$ we get $\left|x_1^{(n)}\right| \leq M$, so $(x_1^{(n)})$ is bounded, and hence by Bolzano-Weierstraß for $\mathbb{R}$ we get that $(x_1^{(n)})$ has a convergent subsequence $(x_1^{n_k})$ that converges to some $x_1 \in \mathbb{R}$.

STEP 2:

Note: We can't just apply Bolzano-Weierstraß to the whole sequence $(x_2^{(n)})$ because that might a priori give us a subsequence $(x_2^{(n_k)})$ for a *different* n_k, which we don't want (it's kind of like getting an toptrain for a different track n_k)

To get around this, consider the subsequence $(x_2^{(n_k)})$ of $x_2^{(n)}$, where n_k is as in $STEP1$. Then since $\left|x_2^{(n)}\right| \leq M$ (by boundedness with $j = 2$), in particular the same is true for $x_2^{(n_k)}$ and therefore $(x_2^{(n_k)})$ is bounded in $\mathbb{R}$ and therefore has a subsequence $(x_2^{(n_{k_l})})$ that converges to some $x_2 \in \mathbb{R}$ as $l \to \infty$ (think of an express-toptrain)



But notice then that $(x_1^{(n_{k_l})})$ is a subsequence of $(x_1^{(n_k)})$ which therefore also converges to x_1 (the toptrain converges, and hence the express-topone converges as well) and hence

$$(x_1^{(n_{k_l})}, (x_2^{(n_{k_l})})) \to (x_1, x_2)$$

STEP 3: Continuing this way at most k times (you can do an inductive construction if you want), we therefore obtain a subsequence of $(x^{(n)})$ with the property that each component converges to some $x_j \in \mathbb{R}$ (for $j = 1, \ldots, n$). So if you let $x = (x_1, \ldots, x_k)$, then that subsequence of $(x^{(n)})$ converges to x $\square$

Let me now give you a taste of topology, exploring the notions of open and closed sets.

6. Open Sets

Video: Open Sets

For this, we first need to define what an open ball is.

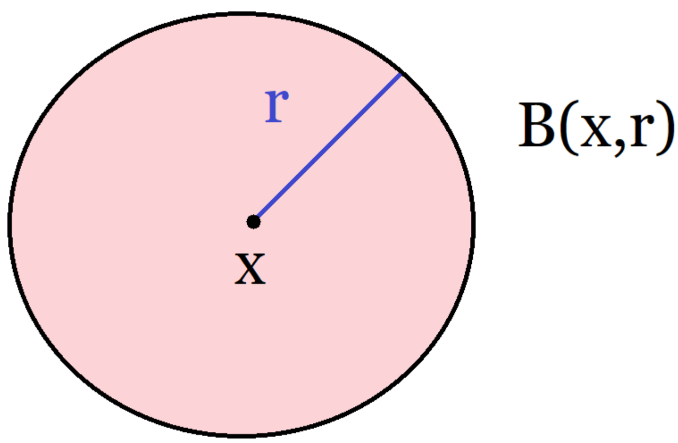
Let (S, d) be a metric space.

Definition:

The open ball centered at x and radius r is:

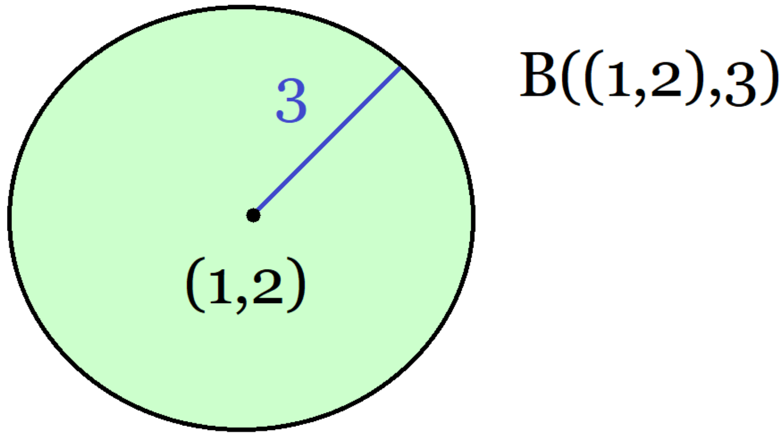
$$B(x, r) = \{y \in S \mid d(x, y) < r\}$$

That is, the set of points that are a distance of at most r away from x.



Example :

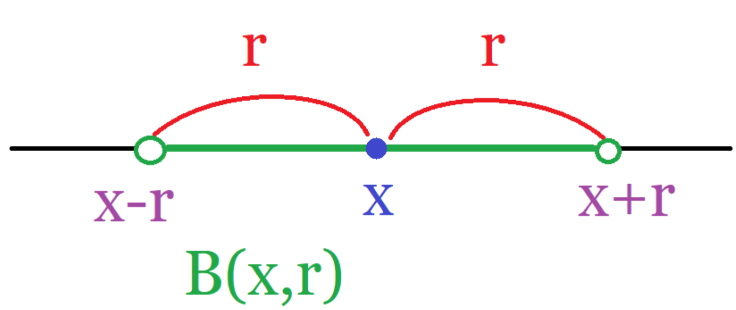
In $\mathbb{R}^2$, $B((1, 2), 3)$ is the open disk centered at $(1, 2)$ and radius 3



Example :

In $\mathbb{R}^2$, $B(x,r) = (x-r, x+r)$

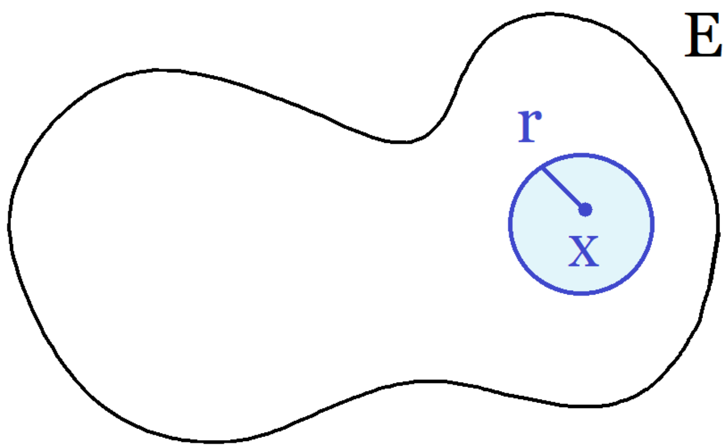
In other words, in $\mathbb{R}$, $B(x,r)$ is just an interval! And this makes sense, because $B(x,r)$ is just the set of points that are at most r away from x



Using this, we can define the concept of an open set:

Definition:

Let E be a subset of S. Then we say E is **open** if for all $x \in E$ there is $r > 0$ such that $B(x, r) \subseteq E$.



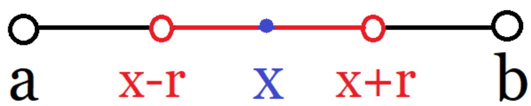
In other words, for every point in E there is some tiny ball that is contained in E.

Interpretation: For every point x in E, you can move around x a little bit and still be in your set. So there is some wiggle room/breathing room around every point.

Example 1:

(a, b) is open

This is because if $x \in (a, b)$, then for some r small enough we have $B(x, r) = (x - r, x + r) \subseteq (a, b)$. In fact, just pick r to be the smaller of $\frac{a+x}{2}$ and $\frac{x+b}{2}$, and you can check that this r works.



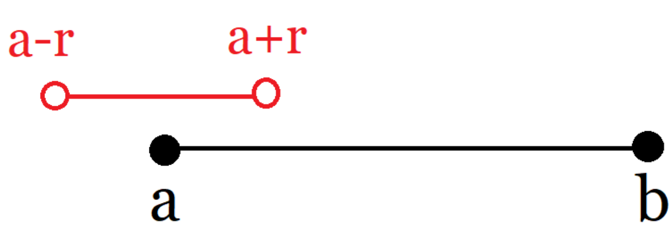
Example 2:

In any metric space, $B(x, r)$ is always open

That's why it's called an open ball[1]

Non-Example 3:

$[a, b]$ is not open.



For $x = a$, no matter how small r is, $B(a, r) = (a - r, a + r) \not\subseteq [a, b]$.

Note: Think of a here as being the edge of a cliff. If you move slightly to the left of a, you've fallen off the cliff. This is not so for open sets:

[1] In case you're curious as to why, here's a proof: If $y \in B(x, r)$, let $r' = r - d(x, y) > 0$, then if $z \in B(y, r')$, then $d(z, x) \leq d(z, y) + d(y, x) < r' + d(x, y) = r - d(x, y) + d(x, y) = r$ so $z \in B(x, r)$ and hence $B(y, r') \subseteq B(x, r)$

No matter what point you're at, you can safely move a bit to the left and to the right.

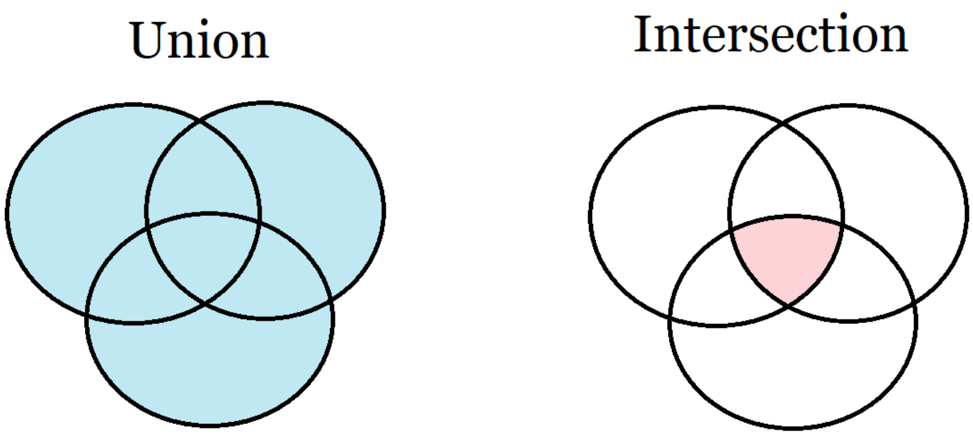
Similarly, $[a, b)$ and $(a, b]$ are not open

Basic Properties:

(1) S and $\emptyset$ are open

(2) The union of *any* collection of open sets is open

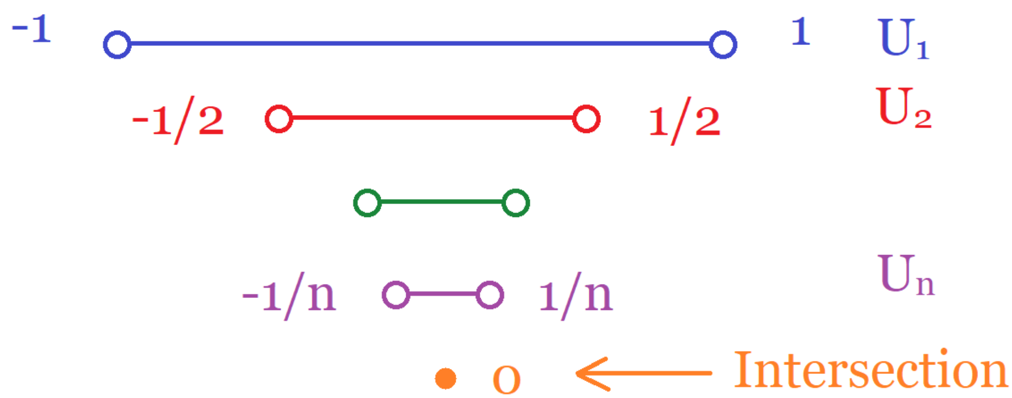
(3) The intersection of finitely many open sets is also open



Warning: The intersection of *infinitely* open sets isn't necessarily open:

Non-Example: Consider $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ in $\mathbb{R}$

So for instance $U_1 = (-1, 1), U_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$ etc.



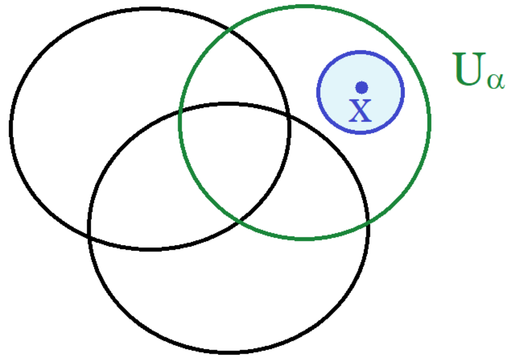
Then each U_n is open, but the intersection of all U_n is $\{0\}$, which is not open.

Note: *Any* set S (not necessarily a metric space) with a collection of open sets satisfying $(1) - (3)$ is called a **topological space**. A metric space is an example of a topological space, but not every topological space is a metric space. The study of topological spaces is called topology. It is a math subject area of its own interest, but also a great ally in analysis; one can prove many interesting results using it. Topology (from *topos logos* = study of places) cares more about the shape of a set rather than distances.

Proof:

(1) Skip (should be immediate)

(2) **Note:** Since the union here isn't necessarily finite (or even countably infinite), we need to do this in full generality: Suppose U is the union of U_α, where each U_α is open.

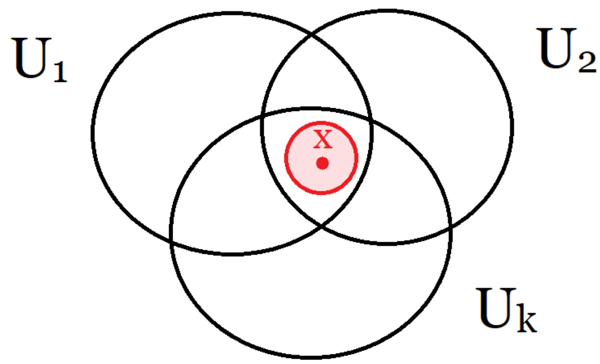


Let $x \in U$, want to show $B(x,r) \subseteq U$ for some $r > 0$.

But by definition of union, we have $x \in U_\alpha$ for some U_α. Since U_α is open, there is some $r > 0$ with $B(x,r) \subseteq U_\alpha \subseteq U$. Therefore $B(x,r) \subseteq U$ ✓

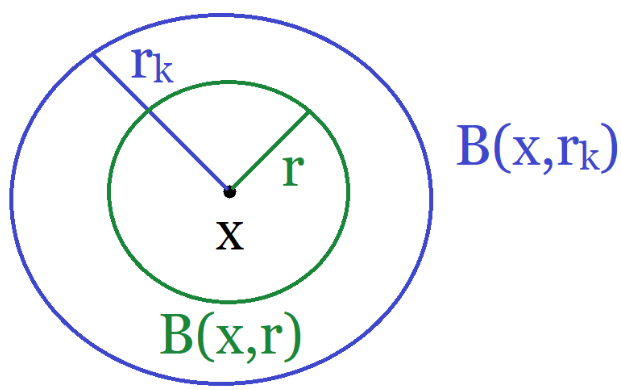
(3) Let U be the intersection of $U_1, U_2, \ldots, U_n$ where each $k = 1, \ldots, n$, U_k is open.

Suppose $x \in U$, need to show there is $r > 0$ such that $B(x,r) \subseteq U$



By definition of intersection, for each k, $x \in U_k$. Therefore, since U_k is open, there is $r_k > 0$ with $B(x, r_k) \subseteq U_k$.

Let $r = \min\{r_1, \ldots, r_n\} > 0$. Then for all k, $r \leq r_k$ and therefore $B(x, r) \subseteq B(x, r_k) \subseteq U_k$.

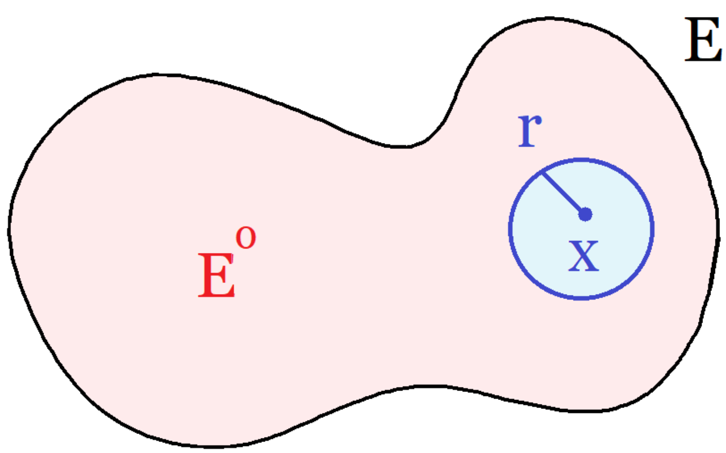


Hence $B(x, r) \subseteq U_k$ for all k, so, by definition of intersection, $B(x, r) \subseteq U$ $\square$

Definition:

We say $x \in E$ is an **interior point** of E if $B(x, r) \subseteq E$ for some $r > 0$.

The set of all interior points of E is denoted by E° (the **interior** of E)

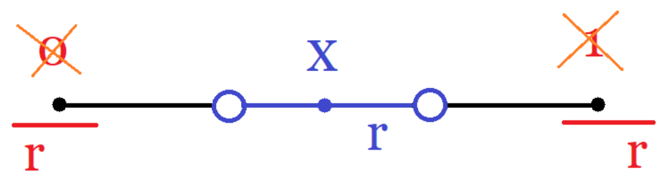


Note: It's similar to the definition of open set except here we're *fixing* a point x. Before, this was true *for all* x.

Example 1:

If $E = [0, 1]$, then $E^\circ = (0, 1)$

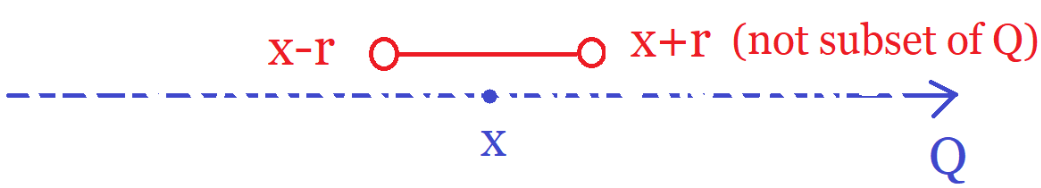
Because for any point *other than* 0 or 1, we can fit a ball inside $[0, 1]$.



Example 2:

If $E = \mathbb{Q}$, then $E^\circ = \emptyset$

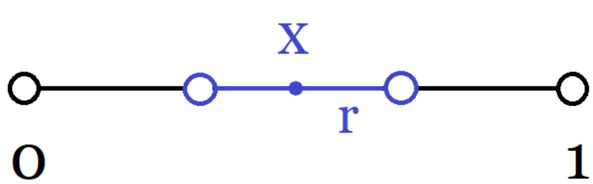
Because for any $x \in \mathbb{Q}$, there is *no* ball $(x - r, x + r)$ that lies entirely within $\mathbb{Q}$. This is because there's always at least one irrational number in $(x - r, x + r)$, no matter what x and r are.



Example 3:

$(0, 1)^\circ = (0, 1)$

Because for *all* $x \in (0, 1)$, there is $r > 0$ with $B(x, r) \subseteq (0, 1)$.



And in fact the above is true for *any* open set:

Fact:

E is open if and only if $E = E^\circ$

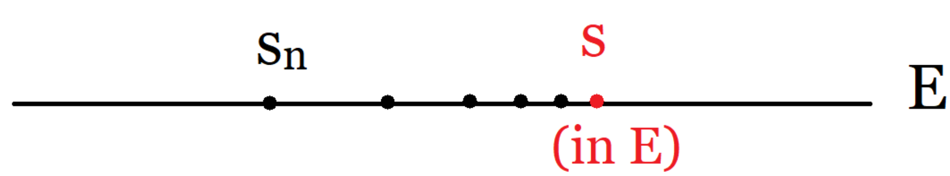
7. Closed Sets

Video: Closed Sets

On the other side of the spectrum comes the notion of a closed set, which has to do with limits of sequences.

Definition:

$E \subseteq S$ is **closed** if, whenever (s_n) is a sequence in E that converges to s, then $s \in E$

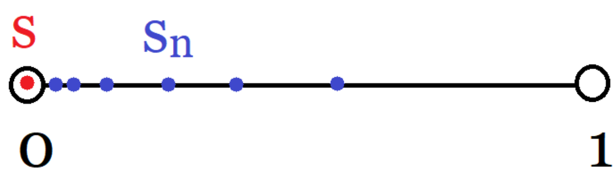


In other words, E must contain all the limits of all the sequences in it. Or, in terms of our analogy, any train in E must have a destination in E.

Non-Example 1:

$(0,1)$ is not closed

For instance, $s_n = \frac{1}{n}$ (with $n \geq 2$) is a sequence in $(0,1)$ that converges to 0, which is not in $(0,1)$.



In some sense, you can escape $(0,1)$ by taking limits, like a prisoner getting out of prison.

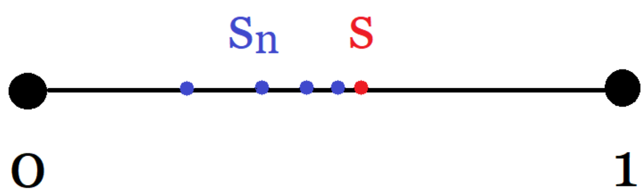
Similarly, $[0,1)$ and $(0,1]$ are not closed.

WARNING: The opposite of 'closed' isn't 'open'! For instance, $[0,1)$ is not closed, but it's not open either! So there are sets that are neither open nor closed, but there are also clopen sets that are both open and closed.

Example 2:

$[0,1]$ is closed

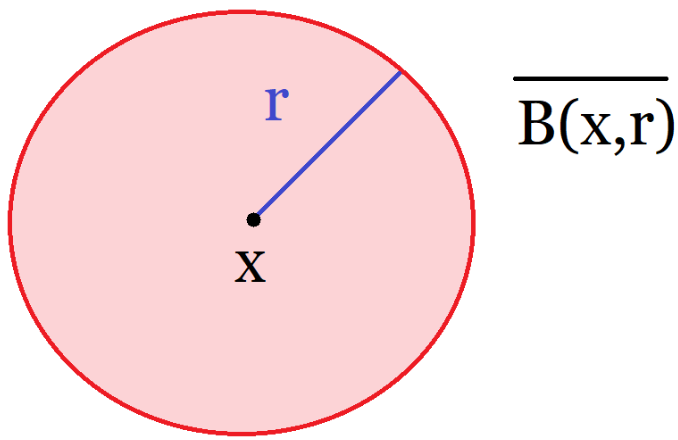
Any convergent sequence in $[0,1]$ must converge to somewhere in $[0,1]$; there is no escaping $[0,1]$, even if we take limits



Example 3:

The *closed* ball is closed

$$\overline{B(x,r)} = \{y \in S \mid d(x,y) \leq r\}$$



Example 4:

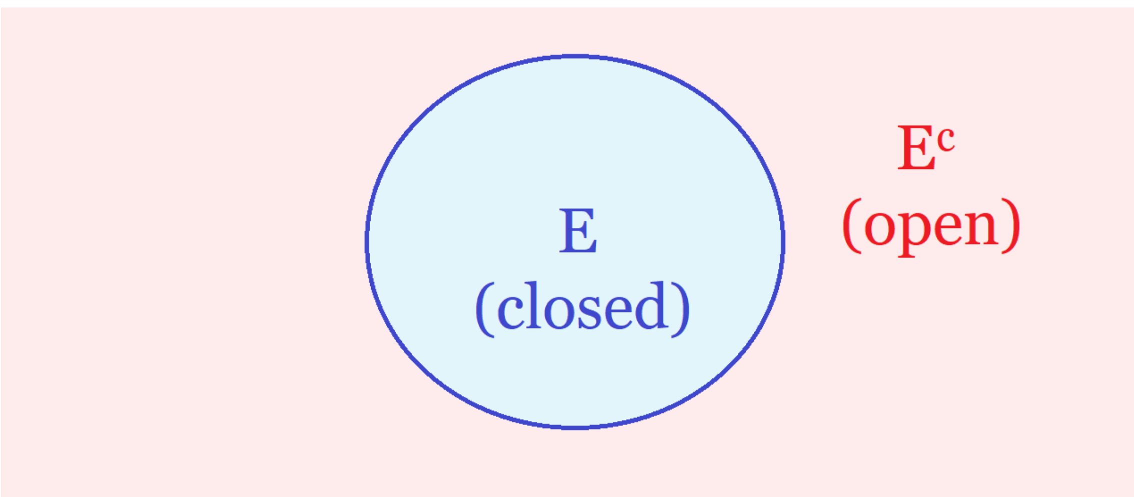
From section 11, if (s_n) is a sequence, then the set S of all limit points (or subsequential limits) of (s_n) is closed

Although it is not quite true that the opposite of closed is open, we do have the following fact:

Fact:

E is closed if and only if E^c is open

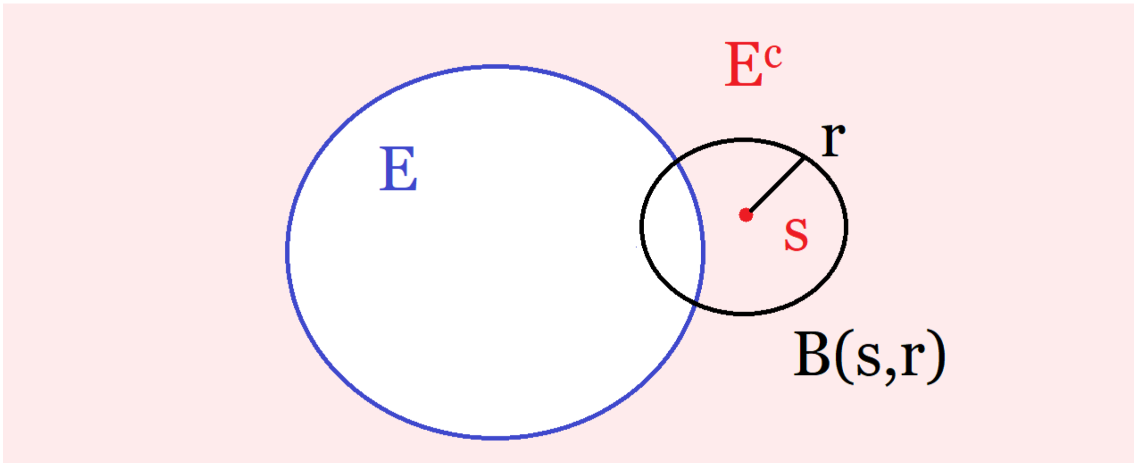
(Recall that E^c is the complement of E, that is $x \in E^c$ if and only if $x \notin E$)



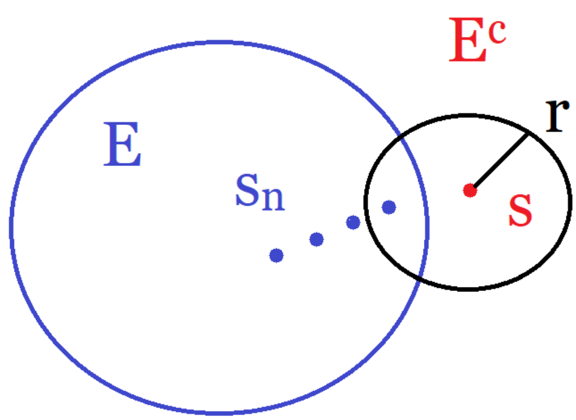
Proof: We will show: E is not closed if and only if E^c is not open.

$(\Rightarrow)$ Suppose E is not closed.

We need to show E^c is not open, that is, there is $s \in E^c$ such that for all $r > 0$, $B(s, r) \not\subseteq E^c$.



Since E is not closed, there is a sequence (s_n) in E with $s_n \to s$ but $s \notin E$, so $s \in E^c$.



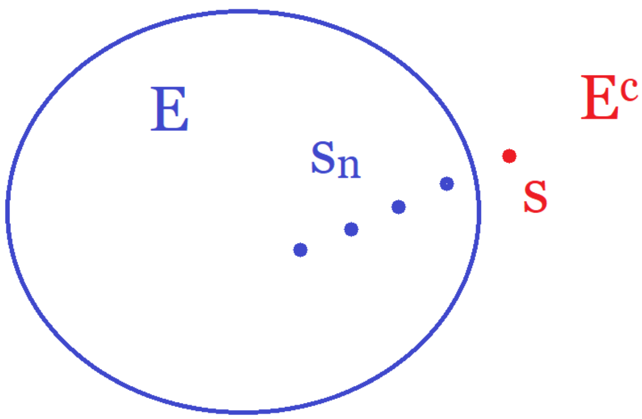
By definition of $s_n \to s$, for every $r > 0$, there is N such that if $n > N$, $d(s_n, s) < r$, which implies $s_n \in B(s, r)$.

But then $B(s, r) \not\subseteq E^c$ since there is an element $s_n \in B(s, r)$ with $s_n \notin E^c$ (because $s_n \in E$).

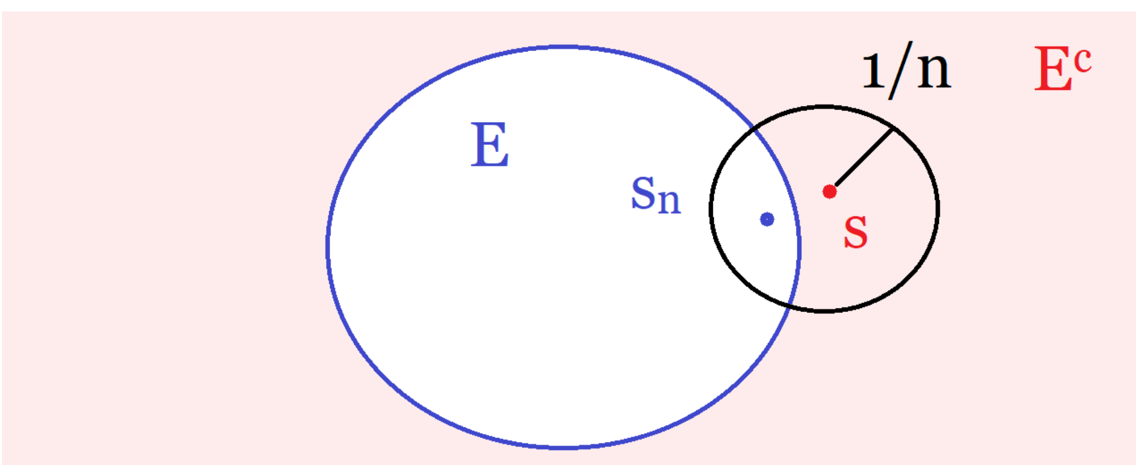
Therefore E^c is not open. ✓

($\Leftarrow$) Suppose E^c is not open.

We need to show E is not closed, that is there is a sequence (s_n) is E with $s_n \to s$ but $s \notin E$.



Since E^c is not open, in particular there is $s \in E^c$ such that for all n, $B\left(s, \frac{1}{n}\right) \not\subseteq E^c$.



Therefore, for every n there is $s_n \in B\left(s, \frac{1}{n}\right)$ (so s_n converges to s) such that s_n is not in E^c, that is $s_n \in E$.

Therefore (s_n) is a sequence in E that converges to s. But since $s \in E^c$, we have $s \notin E$, and therefore E is not closed ✓ □

Just as before, we have the following:

Basic Properties:

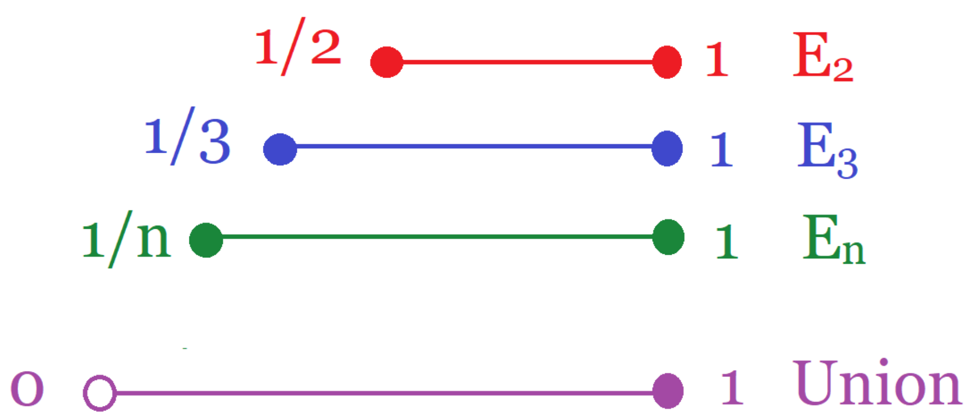
(1) The intersection of any number of closed sets is closed

(2) The union of finitely many closed sets is closed

This actually follows from the analogous statements about open sets, along with the fact that E is closed $\Leftrightarrow$ E^c is open.

WARNING: (2) isn't true if you take infinite unions.

Non-Example: Let $E_n = [\frac{1}{n}, 1]$.



Then the union of E_n is $(0,1]$ (we exclude 0 because it is in none of the E_n above), which is not closed.

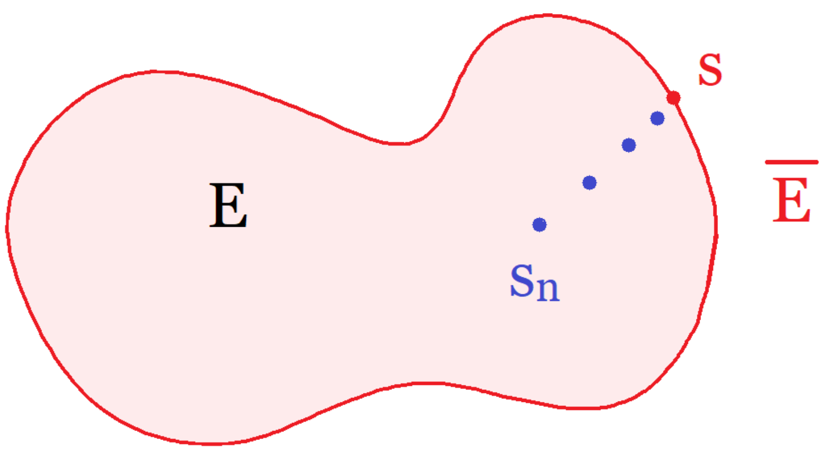
Definition:

We say s is a **limit point** of E if there is a sequence (s_n) in E that converges to s.

The set of all limit points of E is denoted by $\overline{E}$ (the **closure** of E)

Note: The book writes E^- instead of $\overline{E}$.

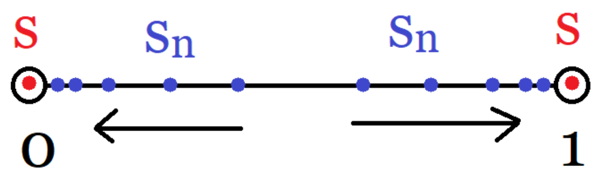
In some sense $\overline{E}$ is the set of all destinations of all trains in E, all the places you can go to.



Example 1:

$\overline{(0,1)} = [0,1]$

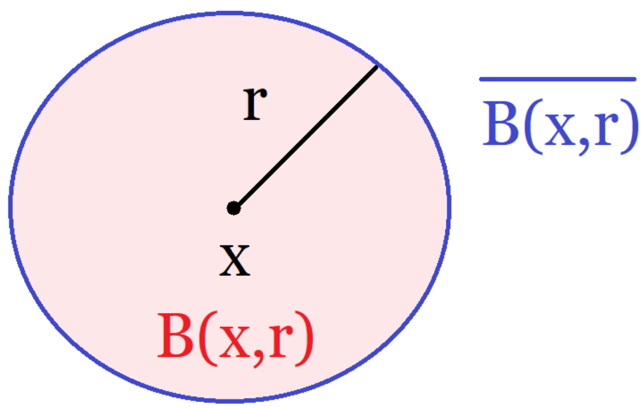
This is because the limit s of any sequence (s_n) in $(0,1)$ is either in $(0,1)$, or is 0, or is 1.



Example 2:

In $\mathbb{R}^k$, the closure of the open ball $B(x,r)$ is the closed ball

$$\overline{B(x,r)} = \{y \in S \mid d(x,y) \leq r\}$$

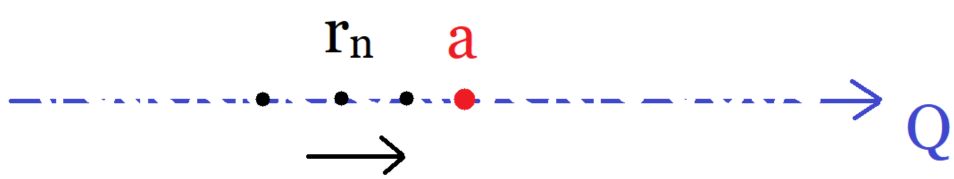


Note: In general, it is not true that the closure of $B(x,r)$ is the same as $\{y \in S \mid d(x,y) \leq r\}$ (see homework), but it is true for $\mathbb{R}^k$. Since we'll be dealing mostly with $\mathbb{R}^k$ anyway, we'll use the notation $\overline{B(x,r)}$ to mean both the closure and the closed ball.

Example 3:

$\overline{\mathbb{Q}} = \mathbb{R}$

Because for every every $a \in \mathbb{R}$ there is a sequence (r_n) of rational numbers that converges to a (see section 11)



Example 4:

$\overline{[0,1]} = [0,1]$

This is because any convergent sequence in $[0,1]$ must converge to somewhere in $[0,1]$.

And in fact, this is true in general, for any closed sets, since *every* point of a closed set must be a limit point.

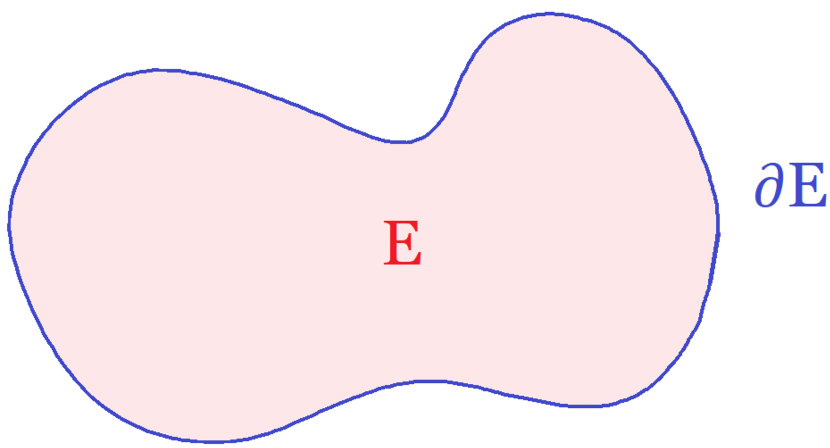
Fact:

Fact: E is closed if and only if $\overline{E} = E$.

Finally, we can define the boundary of E, which is the analog of a skin to a body:

Definition:

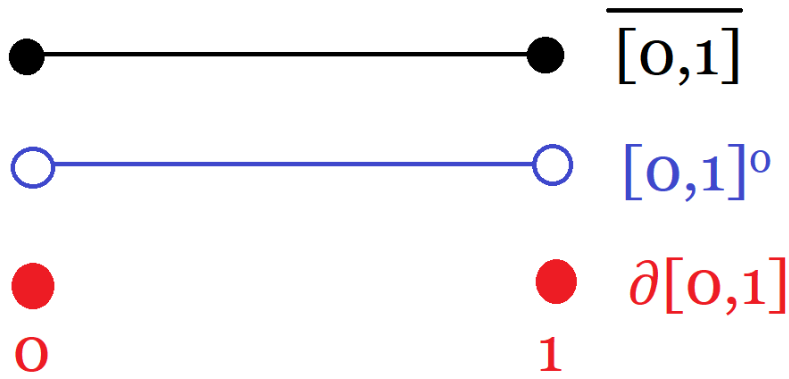
The boundary of E is $\partial E = \overline{E} \backslash E^\circ$



(Makes sense, the skin is the body minus the flesh)

Example 1:

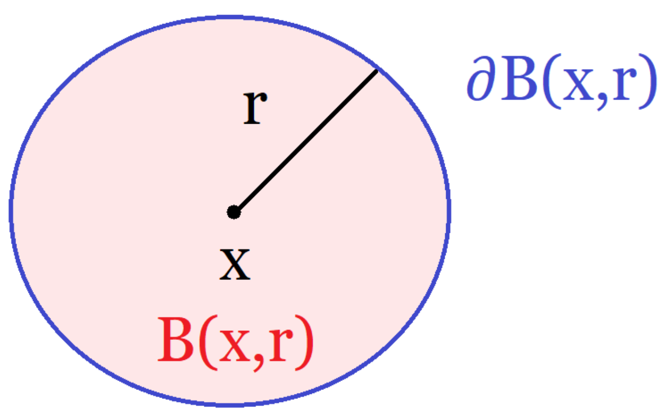
$$\partial[0,1] = \overline{[0,1]} \backslash [0,1]^\circ = [0,1] \backslash (0,1) = \{0,1\} \text{ (Endpoints)}$$



Example 2:

In $\mathbb{R}^k$,

$$\partial B(x,r) = \{y \mid d(x,y) = r\} = \text{ Sphere of radius } r$$



But this is not true in general metric spaces

Example 3:

$$\partial\mathbb{Q} = \overline{\mathbb{Q}} \backslash \mathbb{Q}^{\circ} = \mathbb{R} \backslash \emptyset = \mathbb{R}$$

WOW that's thick!

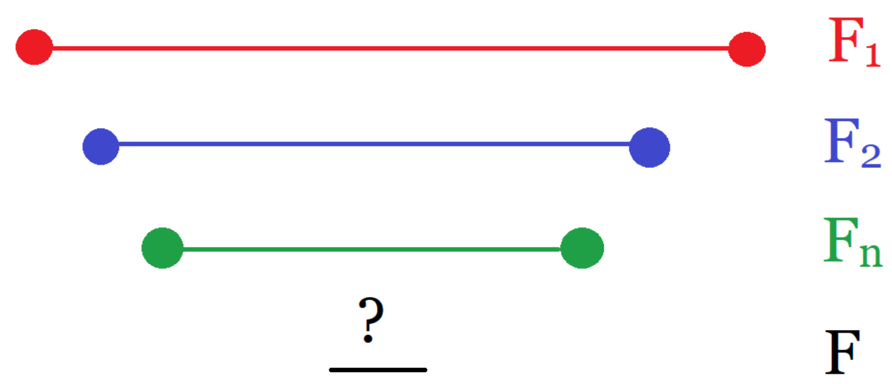
8. Finite Intersection Property

Video: Finite Intersection Property

A very elegant property of closed sets is the **finite intersection property**, which is as follows.

Theorem:

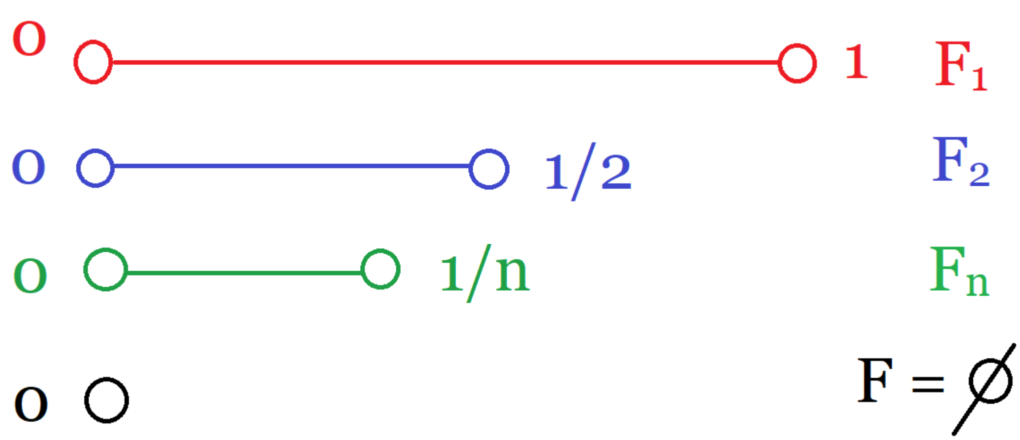
Let (F_n) be a decreasing sequence (meaning $F_1 \supseteq F_2 \supseteq \dots$) of nonempty, closed, and bounded subsets of $\mathbb{R}^k$. Then $F = \bigcap_{n=1}^{\infty} F_n$ is also closed, bounded, and nonempty.



(Think of F_n as nested Russian Matryoshka dolls)

Note: This is **FALSE** if the sets are not closed

Non-Example: Let $F_n = (0, \frac{1}{n})$



Then $\bigcap_{n=1}^{\infty} F_n = \emptyset$, because $\frac{1}{n} \to 0$ and moreover 0 is in none of the F_n.

Remark: This is a very special feature of $\mathbb{R}^k$ and isn't necessarily valid for more general topological spaces. So the proof will use a tool that is very unique to $\mathbb{R}^k$.

Proof: Suppose (F_n) is a decreasing sequence of nonempty, closed, and bounded subsets of $\mathbb{R}^k$ and let $F = \bigcap_{n=1}^{\infty}$.

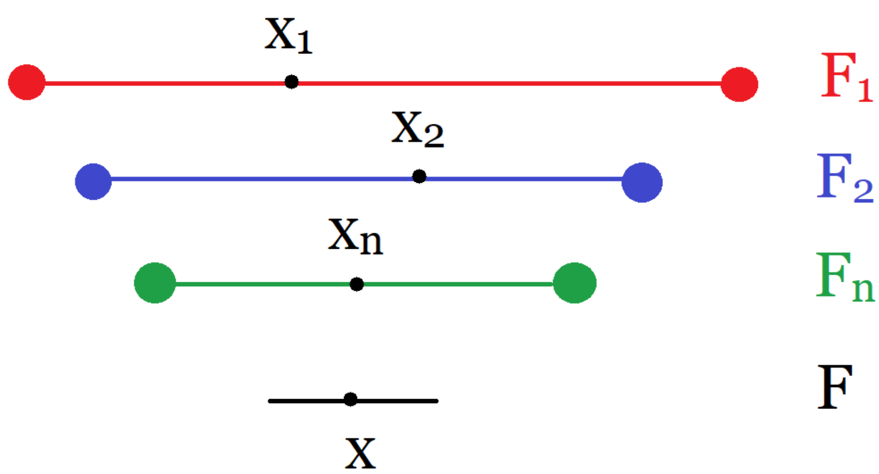
F **is closed:** This is because the intersection of any number of closed sets is closed (see previous section)

F **is bounded:** This is because, $F \subseteq F_1$ and F_1 is bounded by assumption. (see the picture after the theorem)

F **is nonempty:** For each $n = 1, 2, \ldots,$ F_n is nonempty, so let x_n be an element of $\mathbb{F}_n$.

Consider the sequence (x_n). Since for all n, $x_n \in F_n \subseteq F_1$, $x_n \in F_1$ for all n, and since F_1 is bounded (by assumption), then the sequence (x_n) is bounded (in $\mathbb{R}^k$).

Therefore, by the **Bolzano-Weierstrass** (x_n) has a subsequence (x_{n_k}) that converges to some $x \in \mathbb{R}^k$.



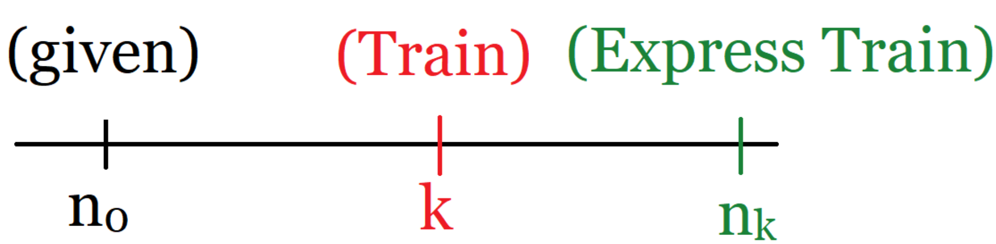
Claim: x is in F

Note: Then we would be done because, since $x \in F$, F is nonempty.

To show $x \in F$, we must show that for all $n_0 \in \mathbb{N}$, $x \in F_{n_0}$.

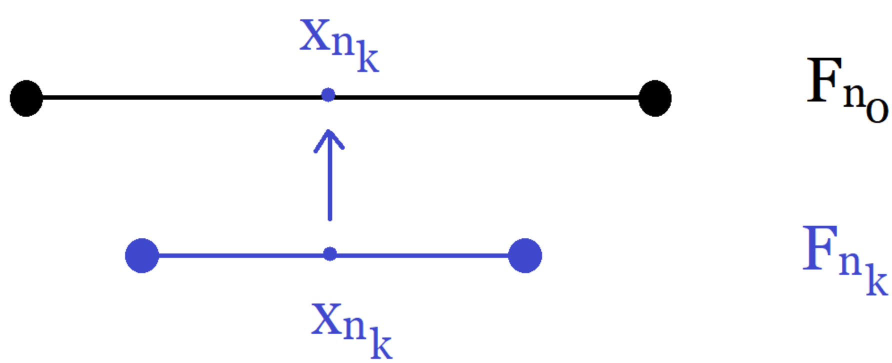
Let n_0 be arbitrary.

Then, for any $k \geq n_0$, we have $n_k \geq n_0$ (since the toptrain is faster than the regular train)

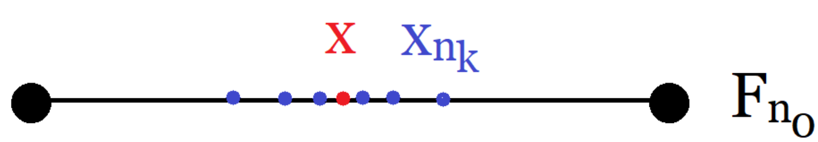


Therefore, $F_{n_k} \subseteq F_{n_0}$ (since sets F_n are decreasing).

Therefore, for every $k \geq n_0$ $x_{n_k} \in F_{n_k}$ (by definition) $\subseteq F_{n_0}$ and so $x_{n_k} \in F_{n_0}$.



But then this means that all the terms of the sequence (x_{n_k}) are eventually in F_{n_0}



Therefore, since F_0 is closed (by assumption) the limit x of (x_{n_k}) is also in F_{n_0}, hence $x \in F_{n_0}$.

Hence, since n_0 was arbitrary we get $x \in F$, so F is nonempty. ✓ $\square$

9. The Cantor Set

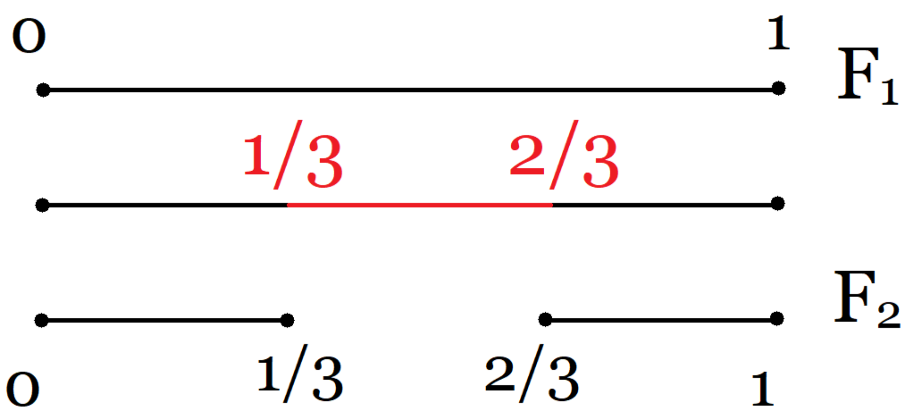
Video: The Cantor Set

Let me quickly introduce you to the single most important set in Analysis: The Cantor Set.

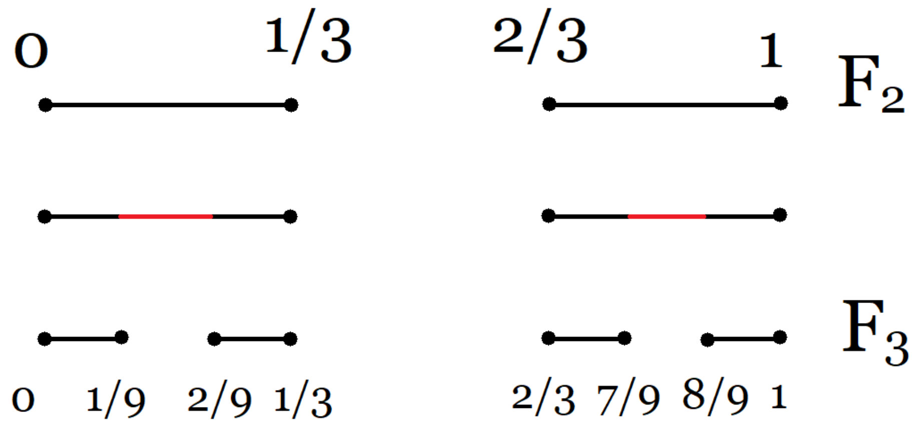
This set is constructed in stages.

STEP 1: Start with $F_1 = [0, 1]$

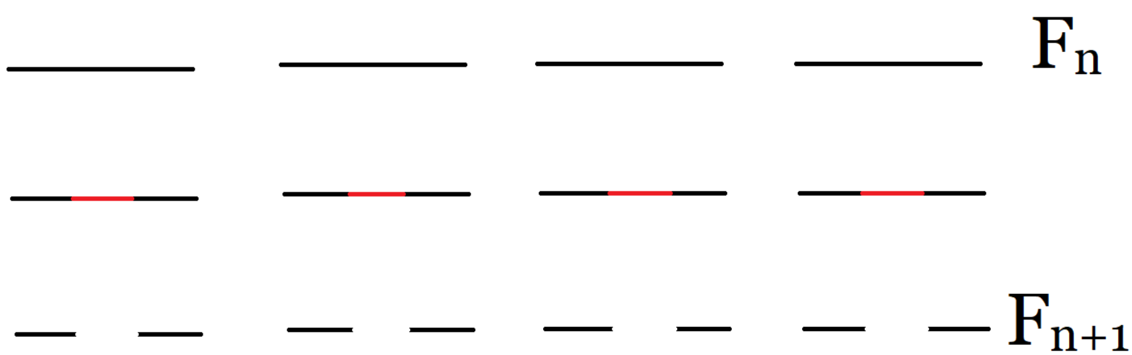
STEP 2: Remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ of F_1 to get two pieces $F_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$



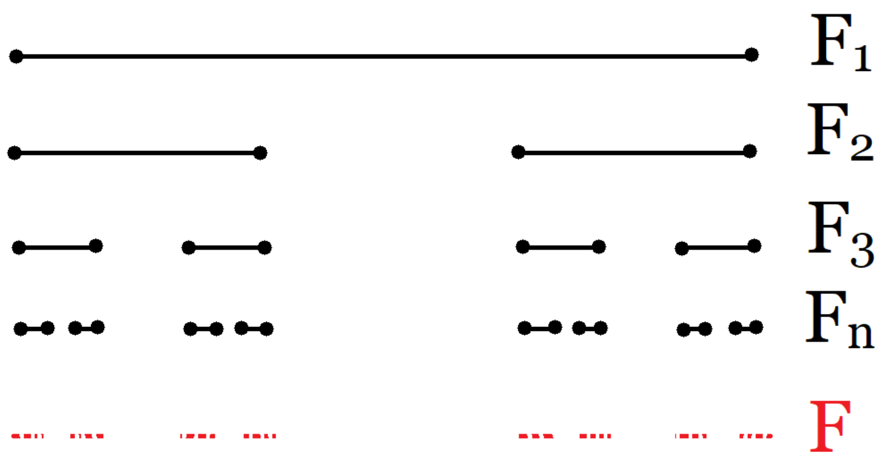
STEP 3: Remove the middle of each piece of F_2 to get 4 pieces $F_3 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$



STEP 4: Continue so on and so forth: Given F_n, define F_{n+1} by removing the middle third of each sub-interval of F_n



And in this way we get a decreasing sequence of intervals $F_1 \supseteq F_2 \supseteq \cdots$



Definition:

The **Cantor set** is

$$F = \bigcap_{n=1}^{\infty} F_n$$

Neat Facts:

The **Cantor set** is

$$F = \bigcap_{n=1}^{\infty} F_n$$

(1) F is closed

(2) F is nonempty

(3) F has size 0

(4) $F^\circ = \emptyset$

(5) F is compact (see below)

(6) F is uncountable

Proof:

(1) F is closed because it is the intersection of closed sets

(2) F is nonempty by the finite intersection property

(3) Since each F_n consists of 2^{n-1} pieces of length $\left(\frac{1}{3}\right)^{n-1}$, each F_n has size $2^{n-1} \times \left(\frac{1}{3}\right)^{n-1} = \left(\frac{2}{3}\right)^{n-1}$ which goes to 0

(4) Suppose $x \in F^{\circ}$, then there is $r > 0$ such that $B(x, r) = (x - r, x + r) \subseteq F$. But since $(x - r, x + r)$ has size $2r > 0$, F must also have size at least $2r$ which contradicts the fact that F has size 0

(5) This is because F is closed and bounded, so by the Heine-Borel Theorem (see below), F is compact

(6) F is uncountable (see below)

Aside: Also F is perfect and totally disconnected (whatever those terms mean). Moreover, it turns out that any metric space can be thought of a subset of the Cantor set.

Discussion: More interestingly, there's a very natural characterization between the Cantor set and ternary expansions (= expansions in base 3):

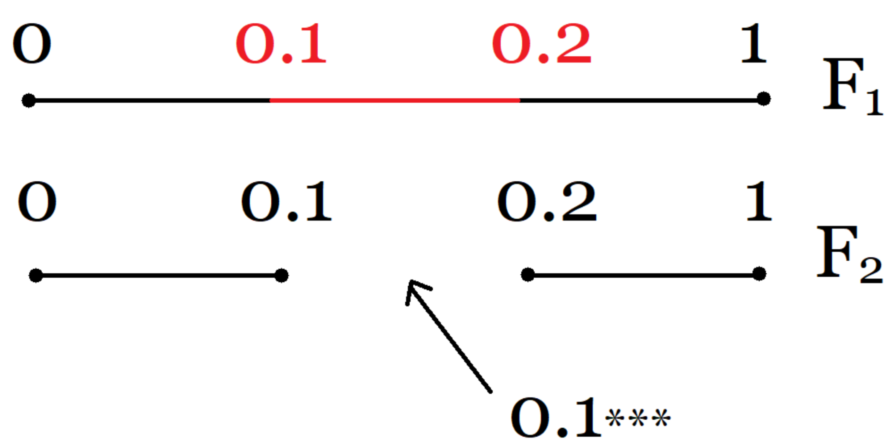
Definition:

Decimal expansion: An expression like $0.1248\ldots$ where you use digits from 0 to 9

Binary expansion: An expression like $0.11011\ldots$ where you use only digits 0 and 1, used a lot in computer science

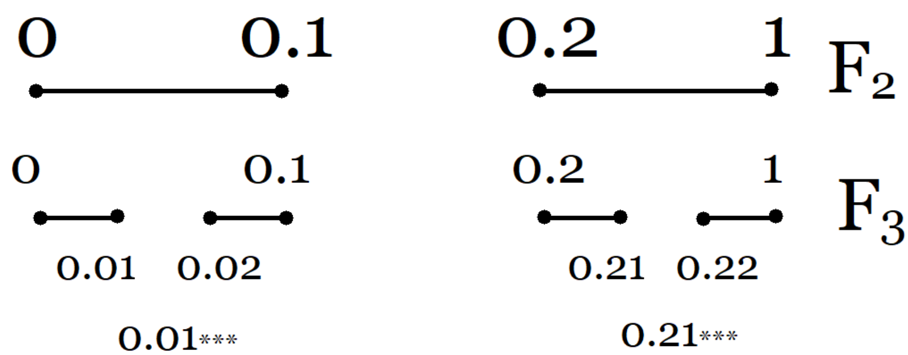
Ternary Expansion: An expression like $0.120211020\ldots$ where you only use digits of $0, 1, 2$

The ternary expansion of $\frac{1}{3}$ is 0.1 and the ternary expansion of $\frac{2}{3}$ is 0.2. So by removing the middle term in F_1 to get F_2, we're essentially throwing away all the numbers of the form $0.1\star\star\star$, so whose first digit is 1



Note: Technically 0.1 is in F2, but you can just write that as $0.022222\cdots$

Similarly, in F_2 we're throwing away numbers of the form $0.01\star\star\star$ and $0.21\star\star\star$, so whose second digit is 1



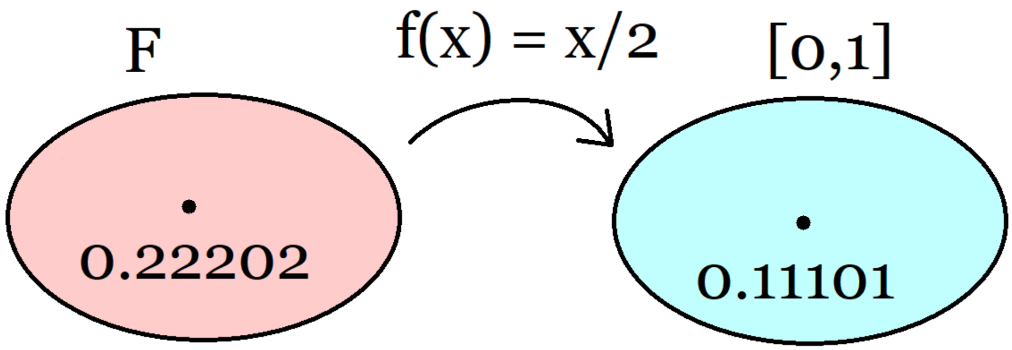
So really, at each step, we're throwing away any number that has a 1 in its expansion, therefore:

Theorem:

F is just the set of all numbers between 0 and 1 that have no 1 in their expansion

Cool Facts:
For instance 0.022022 is in F, but 0.02210202 is not in it.

What makes this super neat is that the function $f(x) = \frac{x}{2}$ turns elements in F into numbers with binary expansions; for instance $f(0.22202) = 0.11101$



Since every element of $[0,1]$ has a binary expansion, f is actually a bijection (= one-to-one correspondence) between F and $[0,1]$

Therefore F has the same cardinality as $[0,1]$, and is hence uncountable!

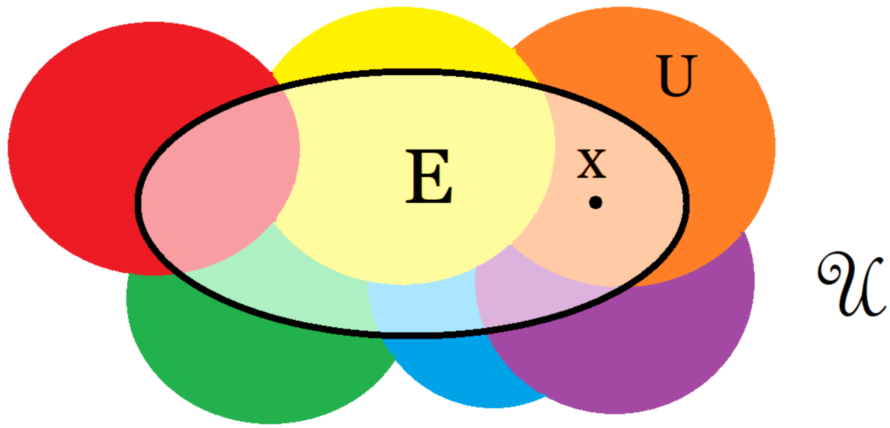
As a grand finale of our metric space extravaganza, let's discuss one of the most powerful concepts in Analysis: Compactness. This notion may seem very abstract at first, but it has very powerful consequences.

10. Covers and Subcovers

Video: Compactness

In order to define compactness, we first need to define the notion of an open cover.

Intuitively: An open cover of E is a family of open sets that *covers* E, as in the picture below. Here E represents the (light) oval region and $\mathcal{U}$ is the collection of all the colored disks U.



Definition:

Let $\mathcal{U}$ be a family of open subsets U of a metric space. Then $\mathcal{U}$ is an **(open) cover** for E if

$$E \subseteq \bigcup \{U \mid U \in \mathcal{U}\}$$

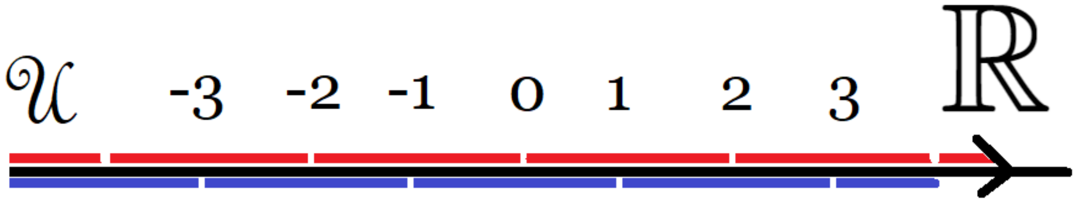
That is: If $x \in E$, then there is some $U \in \mathcal{U}$ with $x \in U$.

What this is saying is that E is included in the union of all the $U's$, so $\mathcal{U}$ literally covers E. Think of the U's as patches that cover the region E.

Example 1:

Let $E = \mathbb{R}$, then the following is an open cover for E:

$$\begin{aligned} \mathcal{U} &= \Big\{(m-1, m+1) : m \in \mathbb{Z}\Big\} \\ &= \Big\{\dots, (-3,-1), (-2,0), (-1,1), (0,2), (1,3), \dots\Big\} \end{aligned}$$

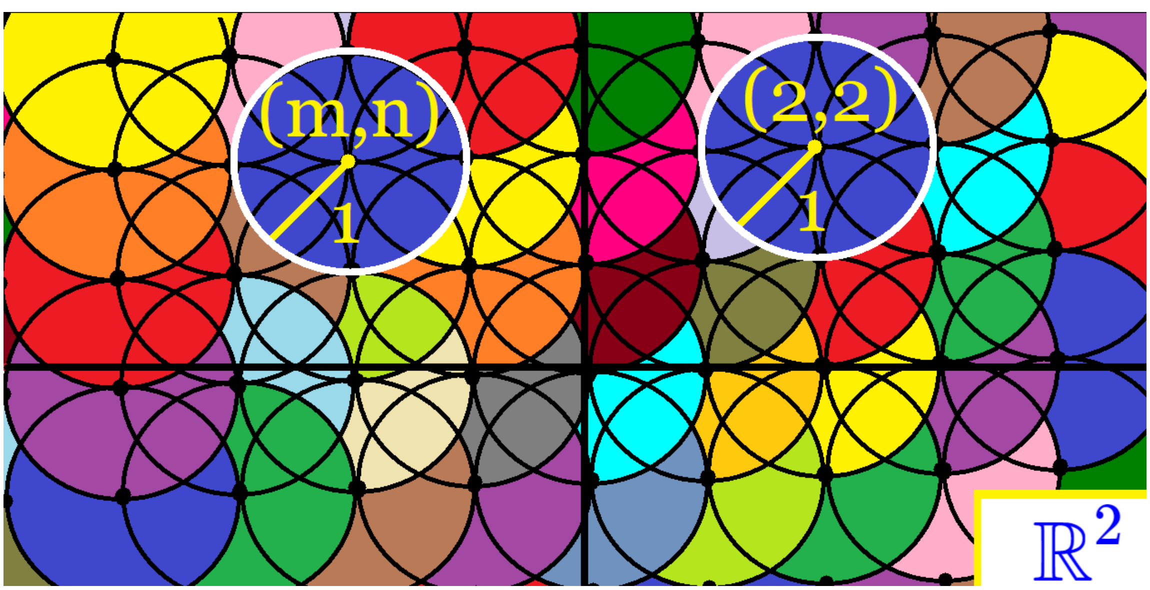


Example 2:

$E = \mathbb{R}^2$

$$\mathcal{U} = \Big\{ B((m,n),1) \,|\, m,n \in \mathbb{Z} \Big\}$$

That is, the family of open balls of radius 1 and centered at pairs of integers covers $\mathbb{R}^2$

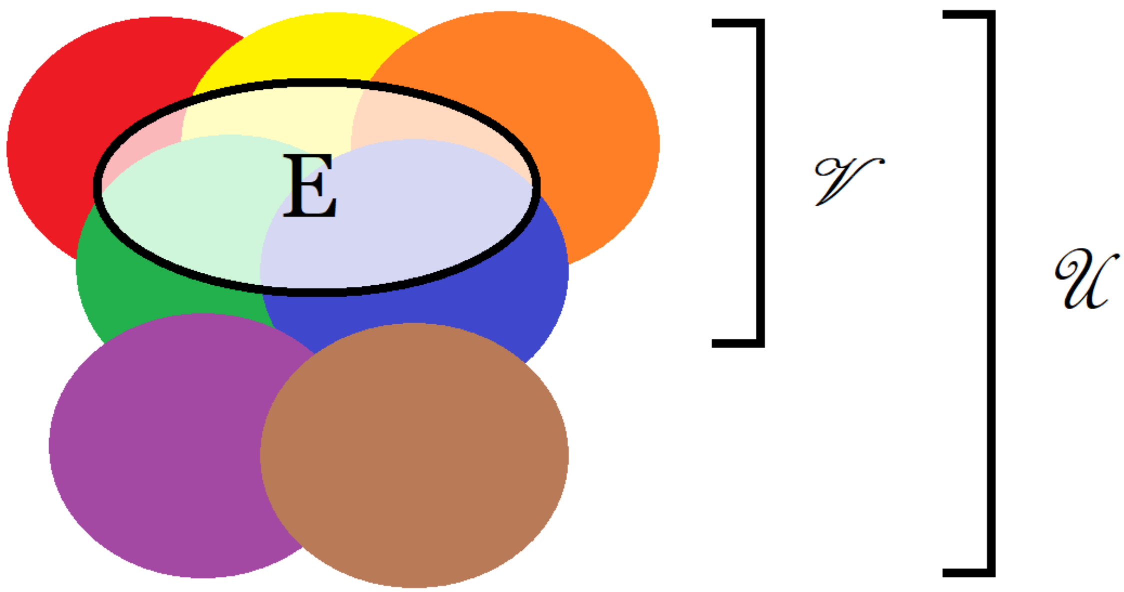


Some covers are better than others! Imagine for instance that each patch U costs \$100 apiece. Then ideally one would like to cover E

with as few patches as possible. This is the idea behind a sub-cover:

Definition:

$\mathcal{V}$ is a **subcover** of $\mathcal{U}$ if $\mathcal{V}$ is a subset of $\mathcal{U}$ that also covers E



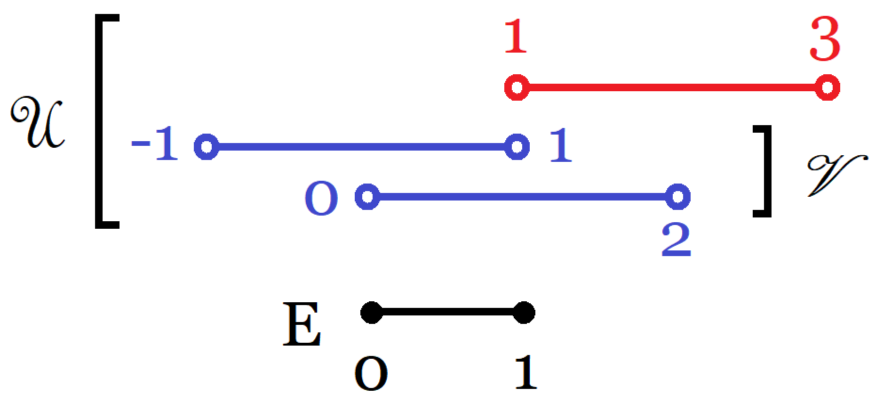
Suppose for instance that, in the picture above, $\mathcal{U}$ consists of all the colored disks, and $\mathcal{V}$ consists of all the disks except for the purple and brown one. Then $\mathcal{V}$ is a subcover of $\mathcal{U}$ since $\mathcal{V}$ is a subset of $\mathcal{U}$ that also covers E

Interpretation: In some sense, $\mathcal{V}$ is better than $\mathcal{U}$. $\mathcal{V}$ still does the job of covering E, but with fewer elements, so it is more cost-efficient

Example:

Let $E = [0, 1]$ and consider

$$\mathcal{U} = \Big\{(-1, 1), (0, 2), (1, 3)\Big\}$$



Then the following is a subcover of $\mathcal{U}$

$$\mathcal{V} = \{(-1, 1), (0, 2)\}$$

Note: In some sense, the $(1, 3)$ in $\mathcal{U}$ is redundant; one wouldn't really pay an extra \$100 for it because $(-1, 1)$ and $(0, 2)$ are already enough to cover E.

Definition:

A **finite** subcover $\mathcal{V}$ is a subcover with finitely many elements

Example: In the above example,

$$\mathcal{V} = \Big\{(-1, 1), (0, 2)\Big\}$$

is a finite subcover of $\mathcal{U}$ because it is a subcover that only has 2 elements, namely $(-1, 1)$ and $(0, 2)$

But

$$\mathcal{V} = \Big\{ \dots, (-2, 0), (-1, 1), (0, 2), (1, 3), (2, 4), \dots \Big\}$$

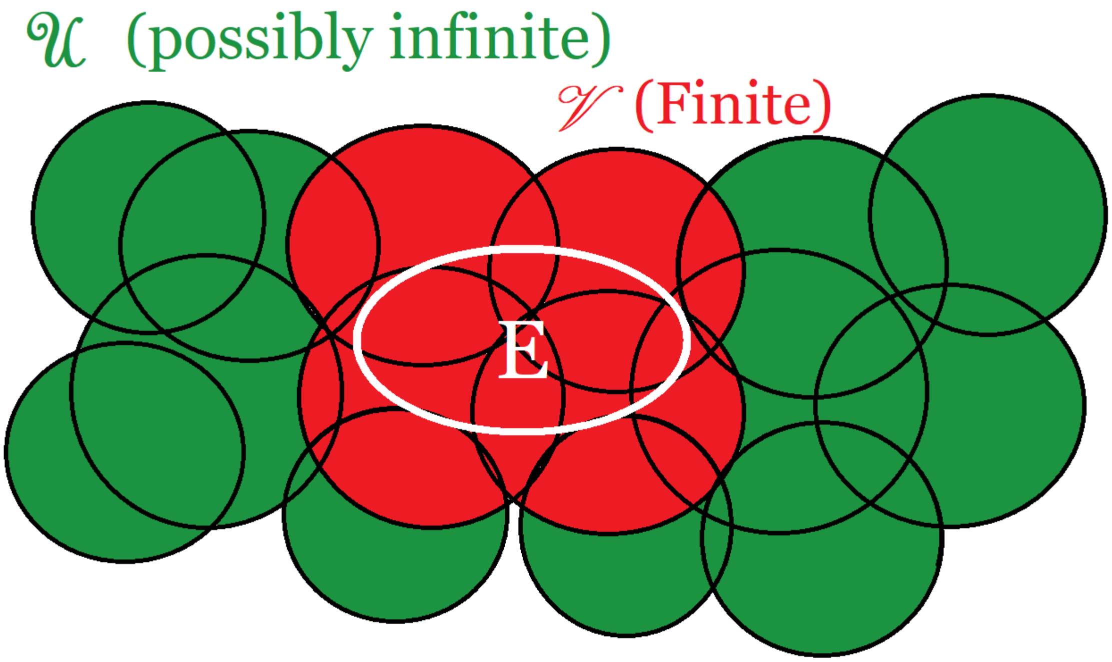
would not be a finite subcover of $\mathcal{U}$ because (1) it has infinitely many elements, and (2) it's not even a subset of $\mathcal{U}$!

11. Definition and Examples

We are now ready for the definition of compactness:

Definition:

A set E is **compact** if <u>every</u> open cover $\mathcal{U}$ of E has a finite subcover $\mathcal{V}$



In other words, compact sets are cost-effective: We *never* need infinitely many patches to cover E. There will **always** be a finite sub-cover, no matter how one covers it. Imagine covering an object with a deck of cards; if the object is compact, you can always cover it with all but finitely many cards.

Note: In practice, it's hard to show that a set is compact, because you'd have to show that *every* cover has a finite sub-cover. It is much easier to show that sets are not compact, as in the examples below. That said, at the end of today we'll find a very elegant criterion to show that a set is compact.

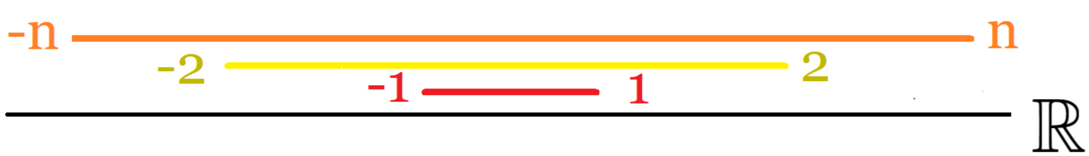
Non-Example 1:

$E = \mathbb{R}$ is not compact.

All we need to do is to find *one* cover that doesn't work; that is *one* cover that does not have a finite subcover.

Consider the following cover $\mathcal{U}$ of $\mathbb{R}$:

$$\mathcal{U} = \left\{(-1,1),(-2,2),\dots\right\} = \left\{(-n,n) : n \in \mathbb{N}\right\}$$

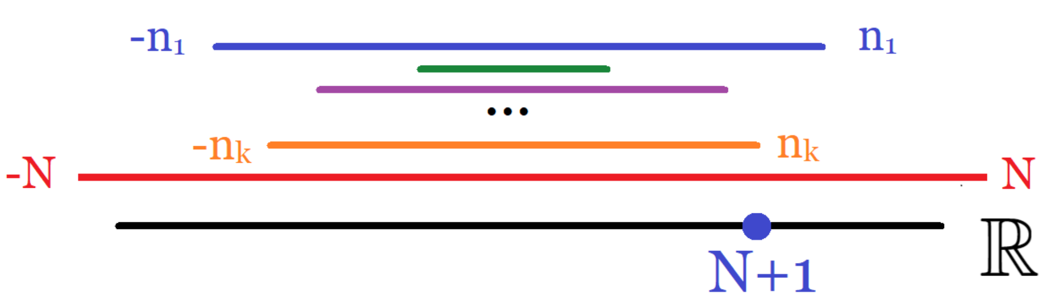


Suppose $\mathcal{U}$ had a finite sub-cover

$$\mathcal{V} = \left\{(-n_1,n_1),\dots,(-n_k,n_k) : n_i \in \mathbb{N}\right\}$$

Let $N = \max\{n_1,\dots,n_k\}$

Then union of $\mathcal{V}$ is $(-N,N)$.



But since $\mathcal{V}$ covers $\mathbb{R}$, $\mathbb{R}$ must be contained in the union of $\mathcal{V}$, so $\mathbb{R} \subseteq (-N,N)$, which gives a contradiction since $N+1 \in \mathbb{R}$ but

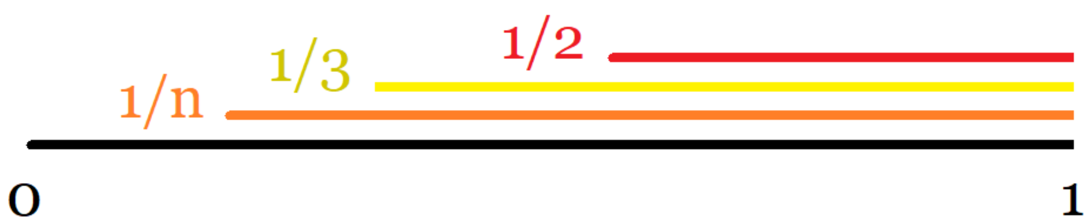
$N+1 \notin (-N, N) \Rightarrow\Leftarrow$. Hence $\mathcal{U}$ has no finite sub-cover, and therefore $E = \mathbb{R}$ is not compact. $\square$

Non-Example 2:

$E = (0, 1)$ is not compact.

Consider the following cover $\mathcal{U}$ of $(0, 1)$:

$$\mathcal{U} = \left\{ \left(\frac{1}{n}, 1\right) : n \geq 2 \right\} = \left\{ \left(\frac{1}{2}, 1\right), \left(\frac{1}{3}, 1\right), \left(\frac{1}{4}, 1\right), \dots \right\}$$

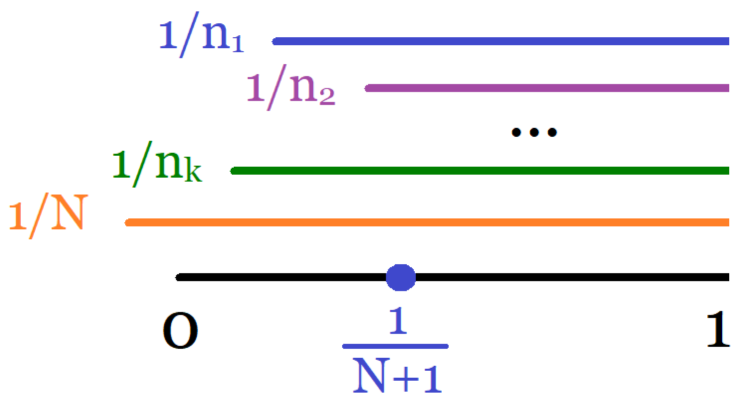


Suppose $\mathcal{U}$ has a finite sub-cover $\mathcal{V}$, where

$$\mathcal{V} = \left\{ \left(\frac{1}{n_1}, 1\right), \dots, \left(\frac{1}{n_k}, 1\right) \right\}$$

Let $N = \max \{n_1, \dots, n_k\} > 0$

Then the union of all the sets in $\mathcal{V}$ is $\left(\frac{1}{N}, 1\right) \neq (0, 1)$ since $x = \frac{1}{N+1} \notin \left(\frac{1}{N}, 1\right)$ even though $x \in (0, 1)$, so $\mathcal{V}$ cannot even cover $E \Rightarrow\Leftarrow$ $\square$



But then what *is* a compact set? This is hard to answer, but luckily, at least in the case of $\mathbb{R}^k$, there's a wonderful theorem called the *Heine-Borel Theorem*, which we'll discuss at the end, that will take care of that.

Example 3:

The following sets are compact in $\mathbb{R}^k$:

(1) Closed intervals $[a, b]$

(2) Boxes like $[1, 2] \times [3, 4]$

(3) Closed balls $\overline{B}(x, r)$ in $\mathbb{R}^k$

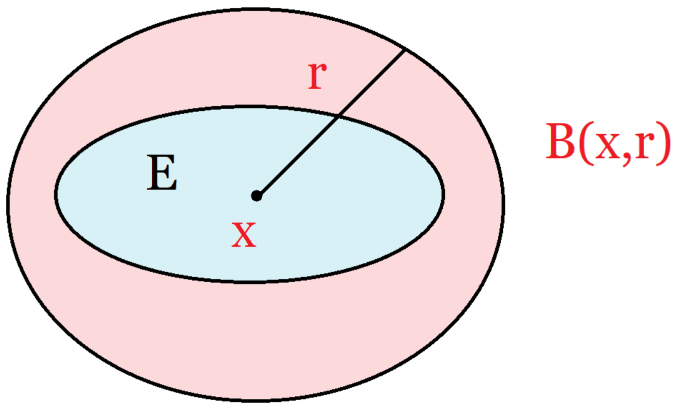
12. Properties of Compactness

Video: Compactness Properties

In this section, we show that compact sets enjoy some nice properties, namely that they must be closed and bounded.

Definition

A (nonempty) set E is **bounded** if there is $x \in E$ and $r > 0$ such that $E \subseteq B(x, r)$.



That is, E is included in some large ball.

Non-Example:

$\mathbb{R}^k$ is not bounded since there's no way to fit all of $\mathbb{R}^k$ inside a ball.

Fact 1:

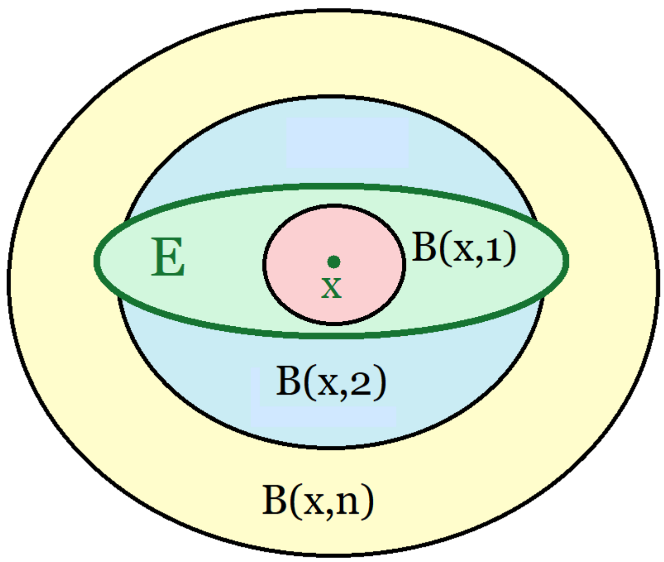
If E is compact, then E is bounded

Note: This is yet another reason why $\mathbb{R}$ (or $\mathbb{R}^k$) are not compact.

Proof: Suppose E is compact. If $E = \emptyset$ then we're done, so assume $E \neq \emptyset$ and let $x \in E$.

Consider the following cover $\mathcal{U}$ of E:

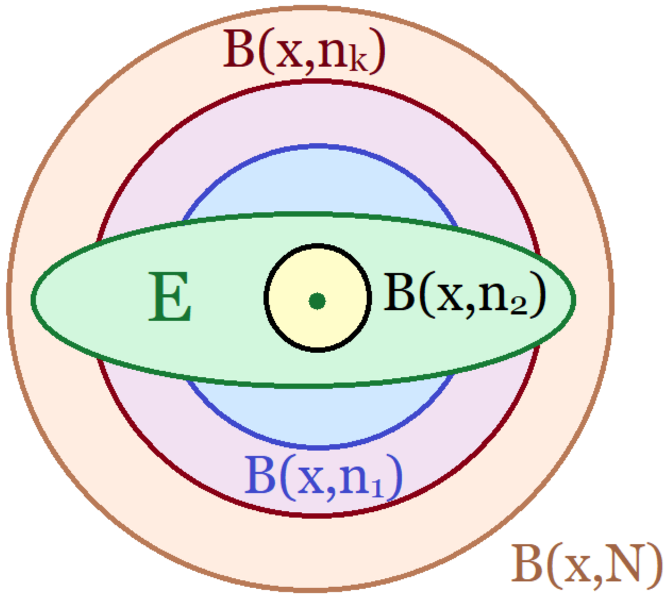
$$\mathcal{U} = \{B(x, n) \mid n \in \mathbb{N}\}$$



Since E is compact, $\mathcal{U}$ has a finite sub-cover:

$$\mathcal{V} = \left\{B(x, n_1), \ldots, B(x, n_k) : n_i \in \mathbb{N}\right\}$$

Let $N = \max\{n_1, \ldots, n_k\} > 0$



Then, on the one hand, the union of $\mathcal{V}$ is $B(x, N)$.

On the other hand, because $\mathcal{V}$ is a sub-cover, that union must contain E, and therefore $E \subseteq B(x, N)$.

But then we found $r > 0$ such that $E \subseteq B(x, r)$, namely $r = N$, and hence E is bounded $\square$

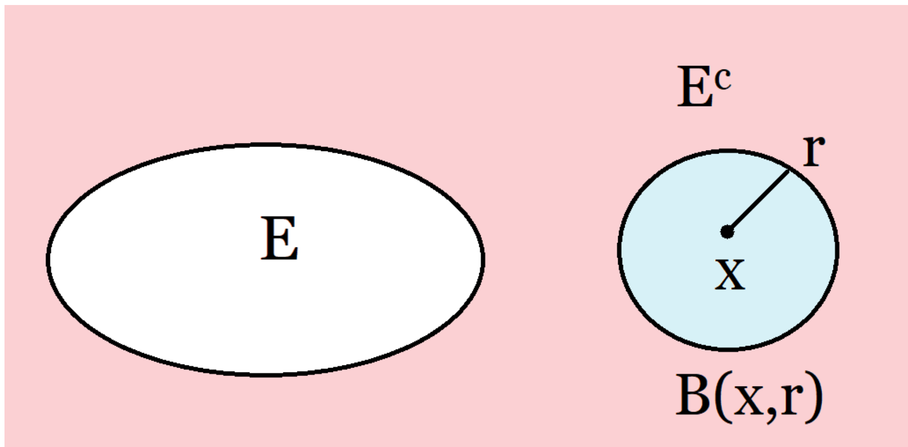
Fact 2:

If E is compact, then E is closed

Note: This is yet another reason why $(0, 1)$ is not compact; it is not even closed.

Proof: To show E is closed, it is enough to show that E^c is open.

Suppose $x \in E^c$. In order to show E^c is open, we need to find r such that $B(x,r) \subseteq E^c$.



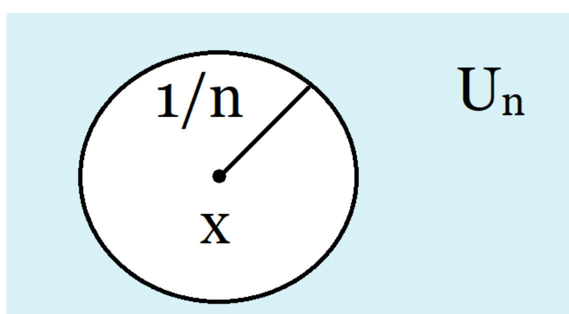
Consider the following clever cover of E:

$$\mathcal{U} = \left\{ U_n \mid n \in \mathbb{N} \right\}$$

Where

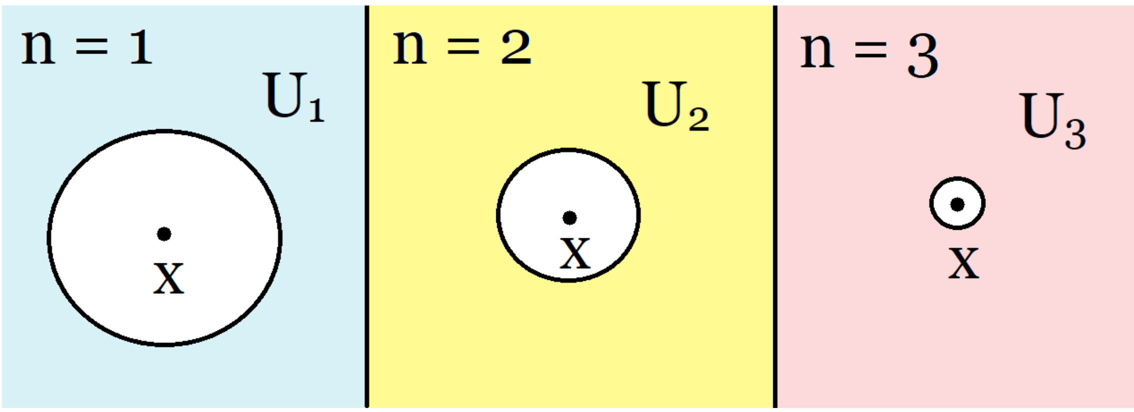
$$U_{n} = \left\{ y \mid d(y,x) > \frac{1}{n} \right\}$$

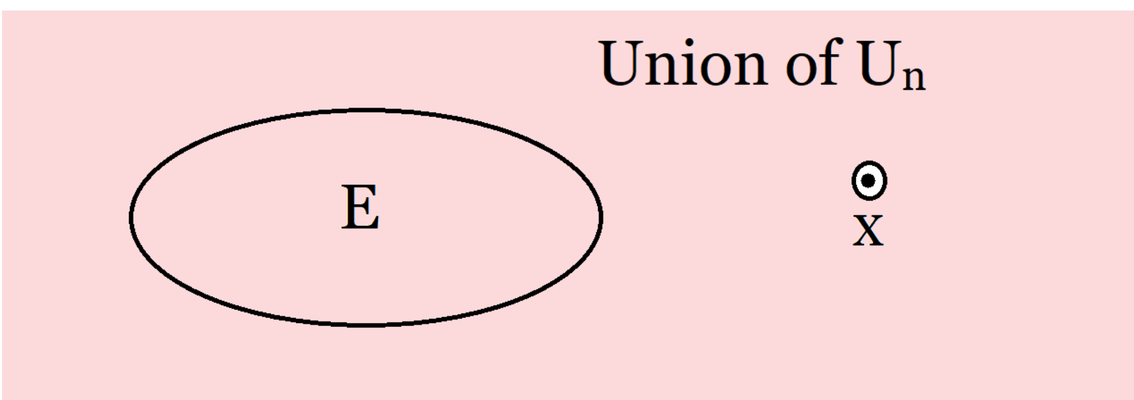
Here U_n consists of all the points that are *at least* $\frac{1}{n}$ away from x. Think of x as being a repellent; no one wants to get close to x



Then, since each $U_n^c = \left\{y \,|\, d(y,x) \leq \frac{1}{n}\right\}$ (which is closed), each U_n is open.

Moreover the union of the U_n is $\{x\}^c$ (see picture below), which therefore covers E given that $x \notin E$.



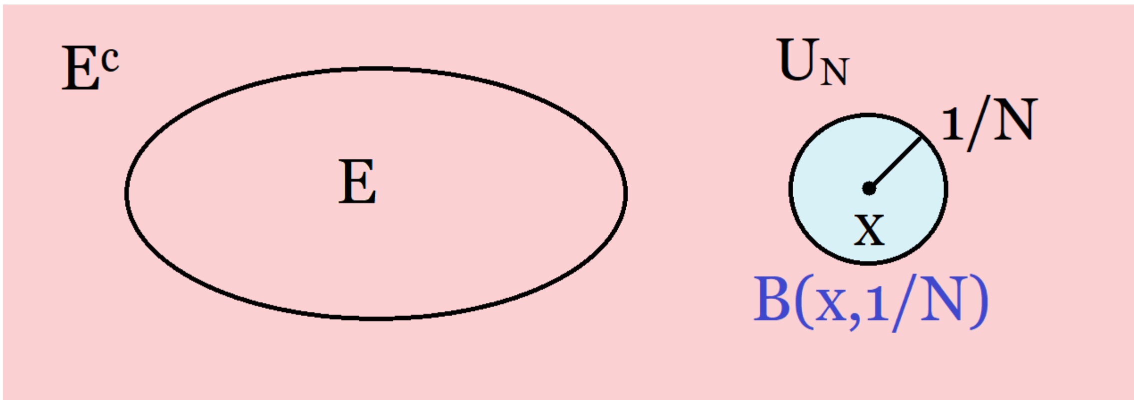


Since E is compact, $\mathcal{U}$ has a finite sub-cover

$$\mathcal{V} = \left\{U_{n_1}, \ldots, U_{n_k} \,|\, n_i \in \mathbb{N}\right\}$$

Let $N = \max\{n_1, \ldots, n_k\} > 0$. Then the union of $\mathcal{V}$ is

$$U_N = \left\{ y \,|\, d(y,x) > \frac{1}{N} \right\}$$



But since $\mathcal{V}$ covers E, it follows that E is included in the union of $\mathcal{V} = U_N$, that is $E \subseteq U_N$. Therefore, taking complements $U_N^c \subseteq E^c$ (see picture above).

But by definition of U_N, we have:

$$U_N^c = \left\{ y \,|\, d(y,x) \leq \frac{1}{N} \right\} \subseteq E^c$$

But then notice that

$$B\left(x, \frac{1}{N}\right) = \{y \,|\, d(y,x) < r\} \subseteq \{y \,|\, d(y,x) \leq r\} \subseteq E^c$$

Hence, if we let $r = \frac{1}{N}$, we get $B(x,r) \subseteq E^c$. This is exactly what we needed to show: We assumed $x \in E^c$ and we needed to find some $r > 0$ such that $B(x,r) \subseteq E^c$.

Hence E^c is open, so E is closed $\square$

13. The Heine-Borel Theorem

Video: Heine-Borel Theorem

From the above, we know that *if* a set is compact, *then* it is closed and bounded. You may ask: Is the converse true? If E is closed and bounded, is it compact? In general, the answer is **NO**, as you'll see on the Homework, or in the following video: Not Compact.

However, in the *special* case of $\mathbb{R}^k$, the following Heine-Borel Theorem says the answer is **YES:**

Heine-Borel Theorem:

A subset E of $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

Examples:

The following subsets are compact, since they are closed and bounded:

(1) Closed intervals $[a, b]$ in $\mathbb{R}$

(2) Boxes like $[1, 2] \times [3, 4]$ (see below)

(3) Closed balls $\overline{B}(x, r)$ in $\mathbb{R}^k$

(4) Spheres in $\mathbb{R}^k$

Warning: As said above, the Heine-Borel theorem only holds for $\mathbb{R}^k$. It is **NOT** true in general!

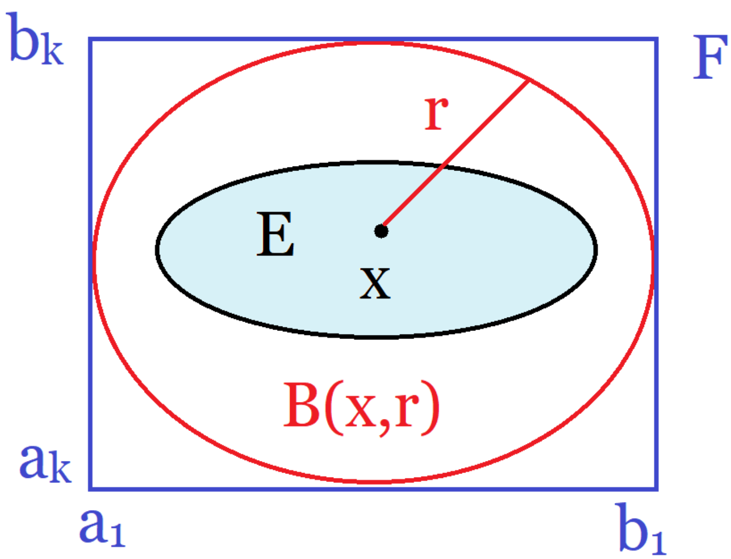
Proof:

($\Rightarrow$) Done above ✓

($\Leftarrow$) Suppose E is closed and bounded.

STEP 1: First of all, since E is bounded, there is $x \in E$ and $r > 0$ large such that $E \subseteq B(x, r)$.

Moreover, every ball is included in a box $F = [a_1, b_1] \times [a_2, b_2] \times [a_k, b_k]$ for some $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$. Namely if $x = (x_1, \ldots, x_k)$, then you can check $a_i = x_i - r$ and $b_i = x_i + r$ works (for $i = 1, \ldots, k$), see picture below



Therefore, we get $E \subseteq B(x, r) \subseteq F$, so $E \subseteq F$, where F is a box

STEP 2: To continue, we need the following fact, which is shown on your homework:

Fact:

A closed subset of a compact set is compact.

Therefore, if we show that the box F is compact, then since E is closed, it would follow that E is compact, and we would be done. ✓

So all that's left to show is the following:

Lemma:

Boxes in $\mathbb{R}^k$ are compact

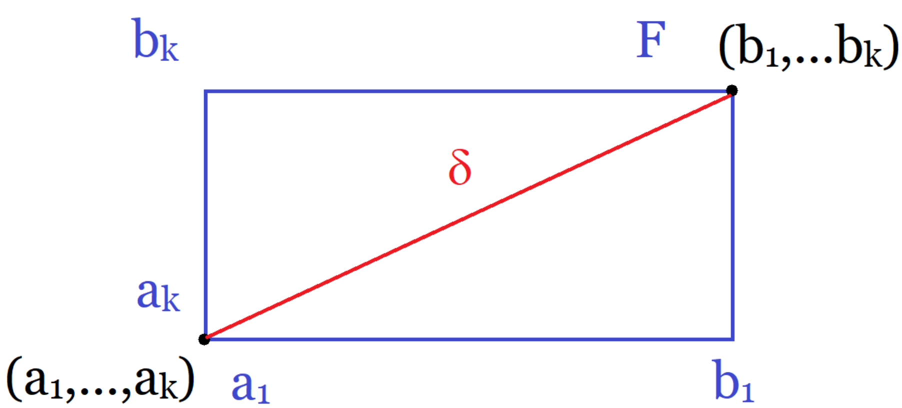
STEP 3: Before we prove this, we need a couple of general properties of boxes. Suppose:

$$F = [a_1, b_1] \times \dots [a_k, b_k]$$

Definition:

The **diameter** of F is

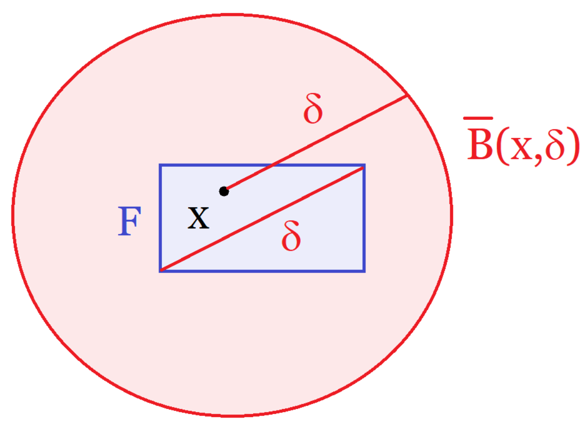
$$\delta = d((a_1, \dots, a_k), (b_1, \dots, b_k)) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_k - a_k)^2}$$



Note that δ is the largest possible distance between points in F, and therefore (here $\overline{B}$ denotes the closed ball)

Remark:

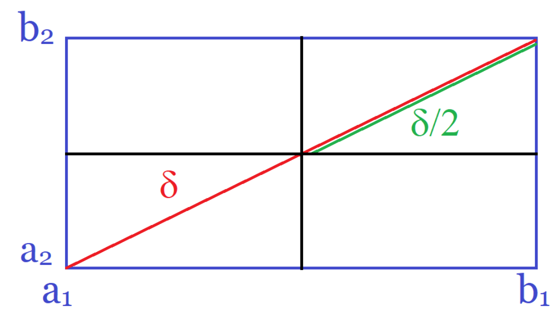
For any $x \in F$, $F \subseteq \overline{B}(x, \delta)$



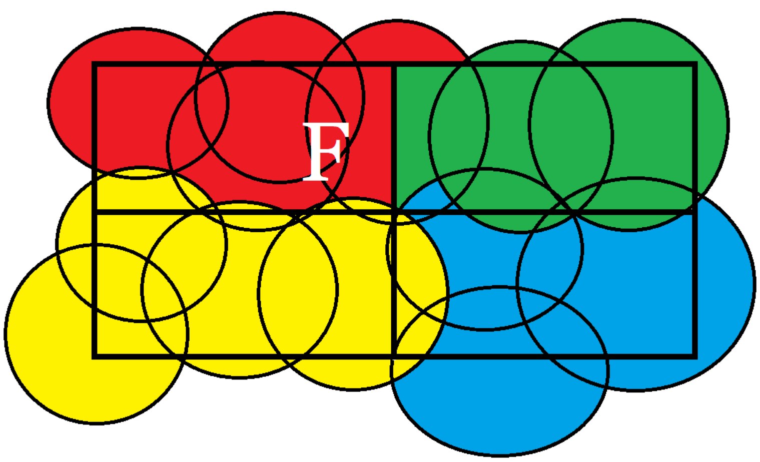
Proof of Lemma: Let $F = [a_1, b_1] \times \cdots \times [a_k, b_k]$ be a box in $\mathbb{R}^k$

STEP 1: Suppose, for sake of contradiction, that F is not compact. Then there is an open cover $\mathcal{U}$ of F that has no finite sub-cover.

The trick here is to split F into 2^k sub-boxes of diameter $\frac{\delta}{2}$ as in the picture below. Here we illustrate the case $k = 2$, where we get $2^2 = 4$ sub-boxes (= quadrants), but in the case $k = 3$ we would get $2^3 = 8$ octants, and in general in $\mathbb{R}^k$ we would get 2^k sub-boxes.

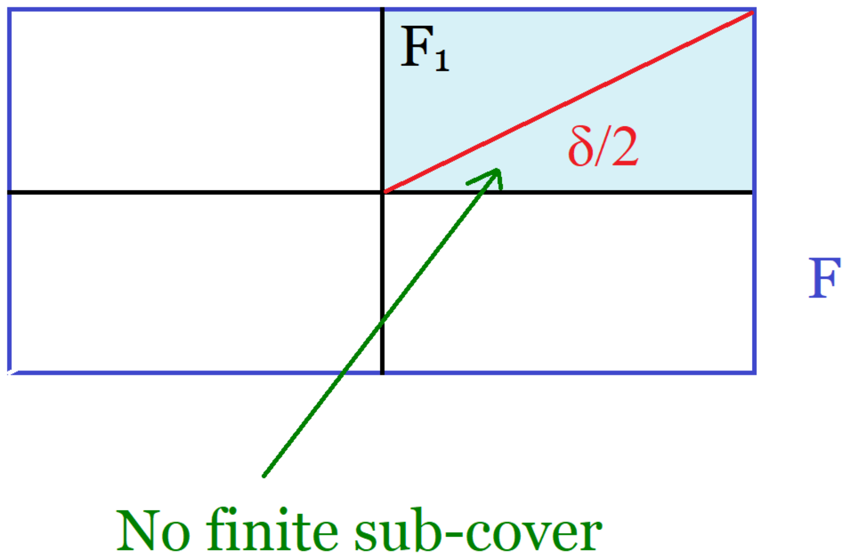


If each of the 2^k sub-boxes of F had a finite sub-cover, then by taking the union of the 2^k finite subcovers, you would get that F has a finite sub-cover $\Rightarrow\Leftarrow$.



4 finite sub-covers = Finite sub-cover for F

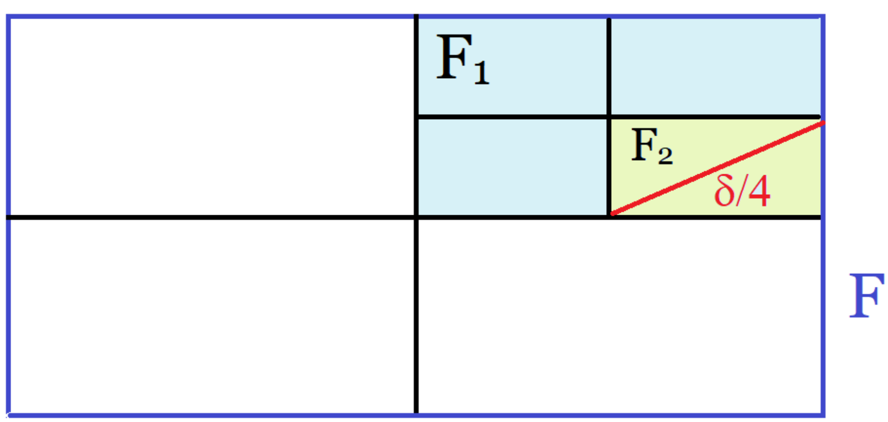
Hence one of the sub-boxes, let's call it F_1 cannot be have a finite sub-cover.



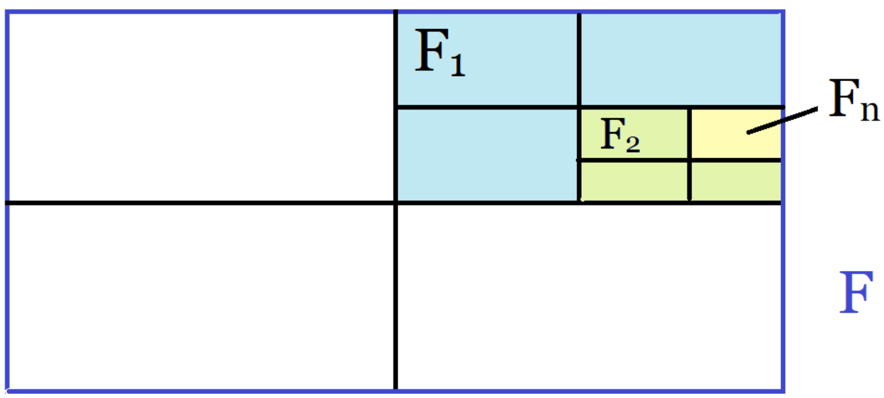
Note that $F_1 \subseteq F$

STEP 2: Now since F_1 is a box of diameter $\frac{\delta}{2}$, F_1 is the union of 2^k (sub-)sub-boxes of diameter $\frac{\delta}{4}$.

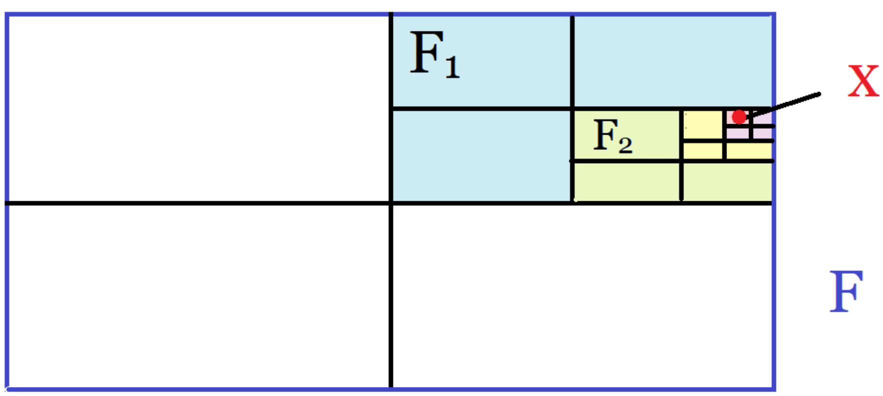
Repeating the argument above, we get there is a (sub-sub) box F_2 of diameter $\frac{\delta}{4}$ that doesn't have a finite sub-cover $\mathcal{U}$. Note that $F_2 \subseteq F_1$



STEP 3: Continuing in this fashion, we obtain a decreasing sequence $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ of sub-boxes such that each F_n has diameter $\frac{\delta}{2^n}$ and that doesn't have a finite sub-cover.

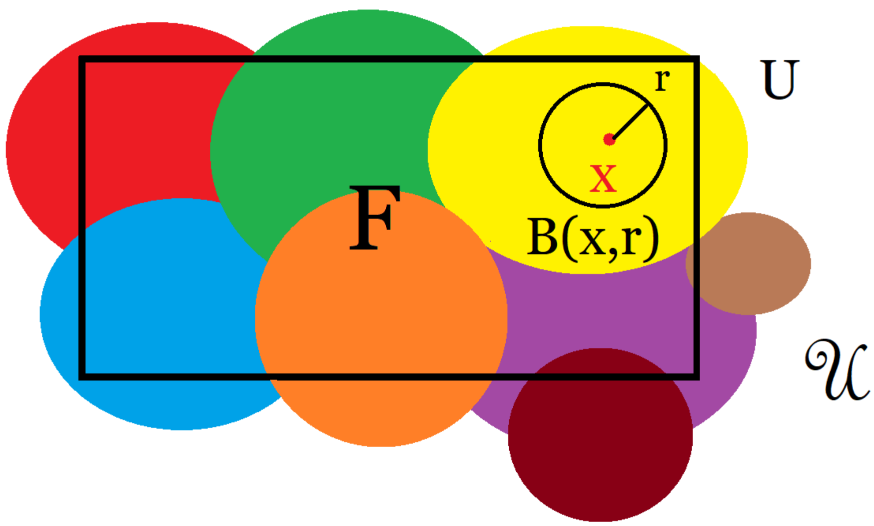


But since each F_n is nonempty, bounded, and closed (since boxes are closed), by the Finite Intersection Property (also known as the Cantor Intersection Theorem), $\bigcap_{n=1}^{\infty} F_n$ is nonempty, so there is $x \in \bigcap_{n=1}^{\infty} F_n$.



STEP 4: Since $\mathcal{U}$ covers F and $x \in F$, there must be $U \in \mathcal{U}$ such that $x \in U$.

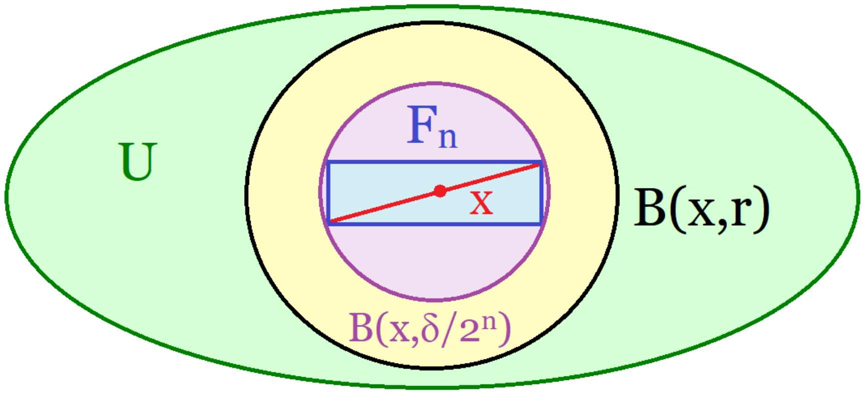
But since U is open, there is $r > 0$ such that $B(x, r) \subseteq U$.



Now let n be large enough so that $\frac{\delta}{2^n} < r$.

Now recall, by the remark (in the green box) preceding this proof, every set is included in the closed ball of radius its diameter. So since $x \in F_n$ (by definition of intersection) and the diameter of F_n is $\frac{\delta}{2^n}$, we get $F_n \subseteq \overline{B}(x, \frac{\delta}{2^n})$, and therefore, since $r > \frac{\delta}{2^n}$, we have

$$F_n \subseteq \overline{B}(x, \frac{\delta}{2^n}) \subseteq B(x, r) \subseteq U$$



Hence $F_n \subseteq U$.

But then this means that the one-element set $\mathcal{V} = \{U\}$ covers F_n, which contradicts the fact that, by construction, F_n doesn't have a finite sub-cover (Remember that we *chose* F_n so that there is no finite sub-cover that covers F_n) $\Rightarrow\Leftarrow$

And this, in turn, contradicts the assumption that $\mathcal{U}$ does not have a finite subcover. Therefore $\mathcal{U}$ *does* have a finite subcover, and so F is compact $\square$

14. Problems

Problem 1:

Definition:

If (S, d) is a metric space, then the (open) ball centered at x and radius r is

$$B(x, r) = \{y \in S \mid d(x, y) < r\}$$

Draw a picture of the unit ball $B((0,0),1)$ (the open ball centered at $(0,0)$ and radius 1) of each of the following metric spaces:

(a) $(\mathbb{R}^2, d_2)$ where $d_2(\mathbf{x},\mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$

(b) $(\mathbb{R}^2, d_1)$ where $d_1(\mathbf{x},\mathbf{y}) = |y_1 - x_1| + |y_2 - x_2|$

(c) $(\mathbb{R}^2, d_\infty)$ where $d_\infty(\mathbf{x},\mathbf{y}) = \max\{|y_1 - x_1|, |y_2 - x_2|\}$

(d) Repeat (a)-(c) but with $\mathbb{R}^3$ instead of $\mathbb{R}^2$

Problem 2:

Find a metric space (S,d) for which the boundary $\partial B(x,r)$ of $B(x,r)$ is **NOT** equal to the sphere of radius r at x, that is $\{y \in S \mid d(x,y) = r\}$.

Problem 3:

Definition:

If (S,d) is a metric space, then we say E is **dense** in S if $\overline{E} = S$ (The book uses E^- instead of $\overline{E}$)

(a) Prove, using the definition above, that $\mathbb{Q}$ is dense in $\mathbb{R}$ (with the usual metric).

Note: You may use the fact from section 11 that for any $a \in \mathbb{R}$, there is a sequence (r_n) of rational numbers converging to a.

Definition:

(S,d) is **separable** if there is a countable subset E of S that is dense in S

(b) Deduce from (a) that $\mathbb{R}$ is separable

(c) Use Lemma 13.3 to show that $\mathbb{R}^k$ is separable

Problem 4: Consider the Cantor set F in Example 5. Use induction on n to show that F_n consists of 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^{n-1}$ Then calculate the total length of F_n, and deduce that the total length of F is 0.

Problem 5: Use the finite intersection property to prove that $[0,1]$ cannot be written as a countably infinite union of disjoint closed subintervals

Problem 6: Show that if (S,d) is a complete metric space and $\{U_n\}_{n=1}^{\infty}$ is a countable collection of open dense subsets of S, then $\bigcap_{n=1}^{\infty} U_n$ is dense in S.

Problem 7: Show that every metric space (S,d) can be completed, just like we completed $\mathbb{Q}$ to get $\mathbb{R}$

Problem 8: Use the definitions below to show the following (do **not** use the definitions in the book)

$$E \text{ is closed if and only if } E^c \text{ is open}$$

Definition:

E is **open** if for all $x \in E$ there is $r > 0$ such that $B(x,r) \subseteq E$
E is **closed** if, whenever (s_n) is a sequence in E that converges to s, then $s \in E$

Problem 9:

Definition:

E is **sequentially compact** if every sequence (s_n) in E has a convergent subsequence

Show that if E is compact, then it is sequentially compact (see hints)

Problem 10:

Definition:

A set E is **totally bounded** if, for every $r > 0$, you can cover E with finitely many balls $B(x, r)$ (where $x \in E$)

Show that if E is compact, then E is totally bounded (this is a one-liner)

Problem 11: Show that if E is totally bounded, then it is separable (= that it has a countable subset F that dense)

Problem 12: Prove *directly* (without Heine-Borel) that $[a, b]$ is compact