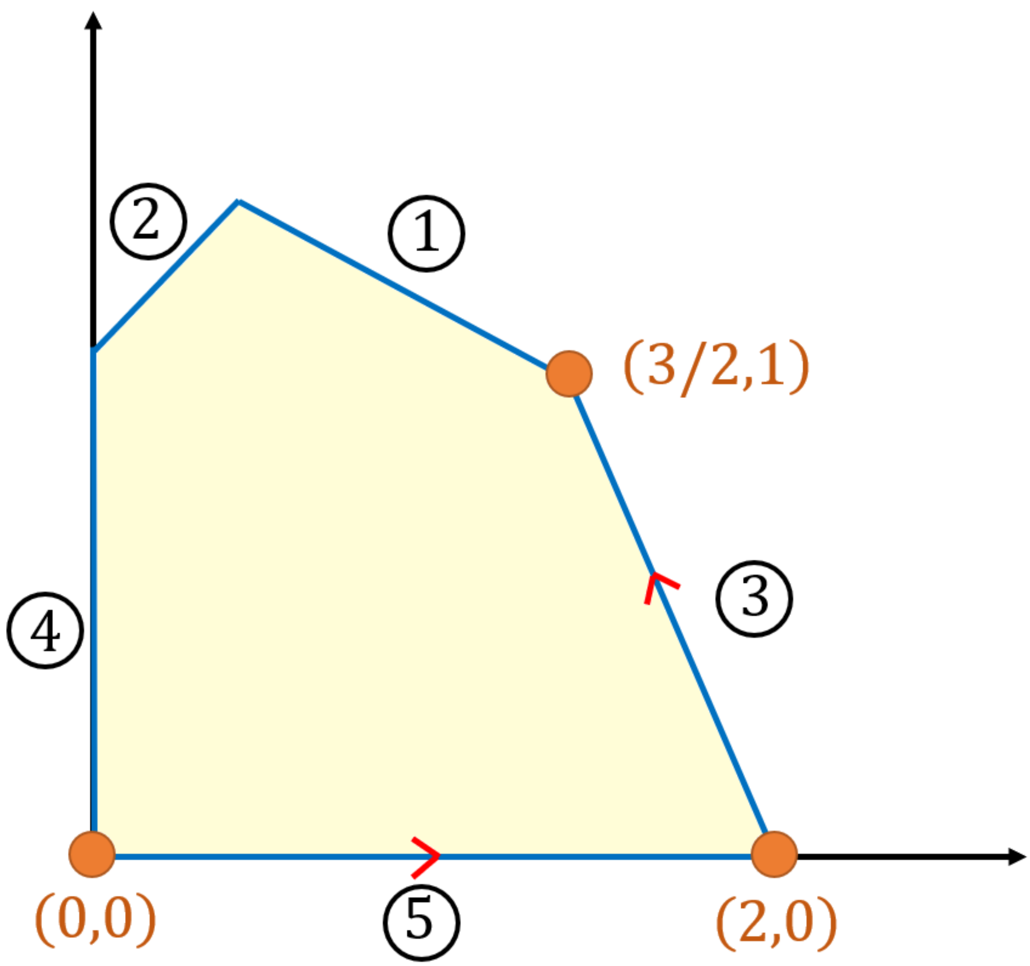
APMA 1210 – FINAL EXAM – SOLUTIONS

1.

$$\begin{aligned} \max z = & 4x_1 + 3x_2 \\ \text{subject to } & 2x_1 + 3x_2 \leq 6 \quad ① \\ & -3x_1 + 2x_2 \leq 3 \quad ② \\ & 2x_1 + x_2 \leq 4 \quad ③ \\ & x_1 \geq 0 \quad ④ \\ & x_2 \geq 0 \quad ⑤ \end{aligned}$$

Optional Picture:



Date: Wednesday, December 14, 2022.

STEP 1: Start at $(0,0)$

Current Vertex: $\{④, ⑤\}$

Objective Value: $z = 0$

Because $4 > 3$, increase x_1 so hold ⑤ and release ④

Hitting times: Here $x_2 = 0$

$$
\begin{aligned}
&① \; 2x_1 + 0 = 6 \Rightarrow x_1 = 3 \\
&② \; -3x_1 + 0 = 3 \Rightarrow x_1 = -1 \times \\
&③ \; 2x_1 + 0 = 4 \Rightarrow x_1 = 2
\end{aligned}
$$

The smallest hitting time is $x_2 = 2$, so ③ is hit first

New Vertex: $\{⑤, ③\} = (2,0)$

Coordinates:

$$
\begin{cases}
③ \; y_1 = 4 - 2x_1 - x_2 \\
⑤ \; y_2 = x_2
\end{cases}
$$

Change coordinates:

$$
\begin{cases}
x_1 = 2 - \frac{1}{2}y_1 - \frac{1}{2}y_2 \\
x_2 = y_2
\end{cases}
$$

Rewrite problem:

$$
\begin{aligned}
\max\ z = &4\left(2 - \frac{1}{2}y_1 - \frac{1}{2}y_2\right) + 3y_2 \\
&① \; 2\left(2 - \frac{1}{2}y_1 - \frac{1}{2}y_2\right) + 3y_2 \leq 6 \Rightarrow 4 - y_1 - y_2 + 3y_2 \leq 6 \\
&② \; -3\left(2 - \frac{1}{2}y_1 - \frac{1}{2}y_2\right) + 2y_2 \leq 3 \Rightarrow -6 + \frac{3}{2}y_1 + \frac{3}{2}y_2 + 2y_2 \leq 3 \\
&③ \; y_1 \geq 0 \\
&④ \; 2 - \frac{1}{2}y_1 - \frac{1}{2}y_2 \leq 4 \Rightarrow -\frac{1}{2}y_1 - \frac{1}{2}y_2 \leq 2 \\
&⑤ \; y_2 \geq 0
\end{aligned}
$$

$$\begin{aligned} \max\ z =& 8 - 2y_1 + y_2 \\ \text{subject to } & -y_1 + 2y_2 \leq 2 \quad \textcircled{1} \\ & \frac{3}{2}y_1 + \frac{7}{2}y_2 \leq 9 \quad \textcircled{2} \\ & y_1 \geq 0 \quad \textcircled{3} \\ & -\frac{1}{2}y_1 - \frac{1}{2}y_2 \leq 2 \quad \textcircled{4} \\ & y_2 \leq 0 \quad \textcircled{5} \end{aligned}$$

STEP 2: $(2, 0)$

Current Vertex: $\{\textcircled{3}, \textcircled{5}\}$

Objective Value: $z = 8$

Because of 1, we increase y_2, so hold ③ and release ⑤

Hitting times: Here $y_1 = 0$

$$\begin{aligned} &\textcircled{2}\ 0 + 2y_2 = 2 \Rightarrow y_2 = 1 \\ &\textcircled{3}\ 0 + \frac{7}{2}y_2 = 9 \Rightarrow y_2 = \frac{18}{7} \\ &\textcircled{5}\ -0 - \frac{1}{2}y_2 = 2 \Rightarrow y_2 = -4 \times \end{aligned}$$

The smallest hitting time is $y_2 = 1$, so ① is hit first

New Vertex: $\{\textcircled{1}, \textcircled{3}\} = (0, 1)$ in $y-$coordinates

Coordinates:

$$\begin{cases} \textcircled{3}\ z_1 = y_1 \\ \textcircled{1}\ z_2 = 2 + y_1 - 2y_2 \end{cases}$$

Change coordinates:

$$\begin{cases} y_1 = z_1 \\ y_2 = 1 + \frac{1}{2}z_1 - \frac{1}{2}z_2 \end{cases}$$

Rewrite problem:

$$z = 8 - 2z_1 + \left(1 + \frac{1}{2}z_1 - \frac{1}{2}z_2\right) = 9 - \frac{3}{2}z_1 - \frac{1}{2}z_2$$

Since both coefficients are negative, we **STOP**

STEP 3: Answer

Optimal $z-$value: $z = 9$

Optimal Vertex:

In $y-$coordinates the vertex is $(0,1)$ and so $y_1 = 0$ and $y_2 = 1$ and so in x-coordinates this becomes

$$\begin{cases} x_1 = & 2 - \frac{1}{2}y_1 - \frac{1}{2}y_2 = 2 - 0 - \frac{1}{2}(1) = \frac{3}{2} \\ x_2 = & y_2 = 1 \end{cases}$$

And so the optimal vertex is $\left(\frac{3}{2}, 1\right)$

Answer: Optimal vertex $\left(\frac{3}{2}, 1\right)$ with $z-$value $z = 9$

2. Let (x_1, x_2, x_3) be Alakazam's strategy and (y_1, y_2) be Bulbasaur's strategy.

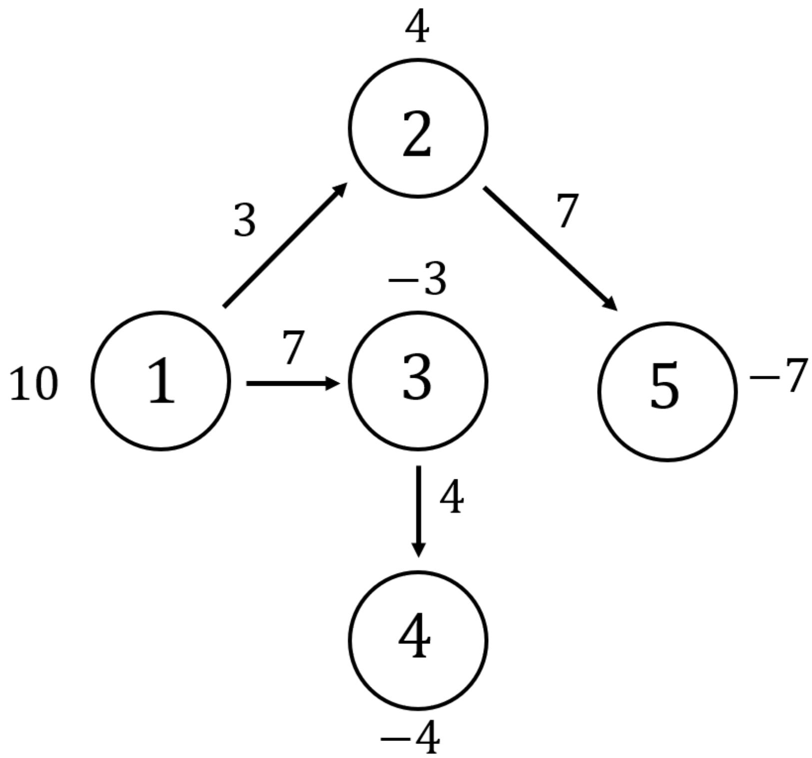
Alakazam's LP Problem:

$$
\begin{aligned}
&\max z\\
\text{subject to } & 10x_1 + 0x_2 - 5x_3 + z \leq 0\\
& 8x_1 - 12x_2 + 10x_3 + z \leq 0\\
& x_1 + x_2 + x_3 = 1\\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Bulbasaur's LP Problem: (Dual Problem)

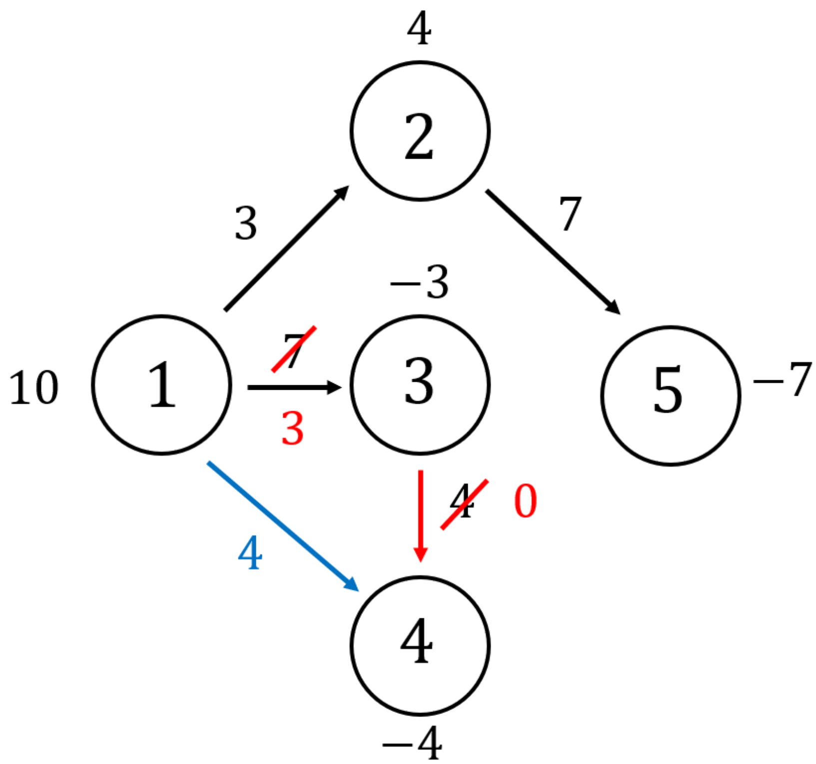
$$
\begin{aligned}
&\min w\\
\text{subject to } & 10y_1 + 8y_2 + w \geq 0\\
& 0y_1 - 12y_2 + w \geq 0\\
& -5y_1 + 10y_2 + w \geq 0\\
& y_1 + y_2 = 1\\
& y_1, y_2 \geq 0
\end{aligned}
$$

3. **STEP 1:** Initial tree

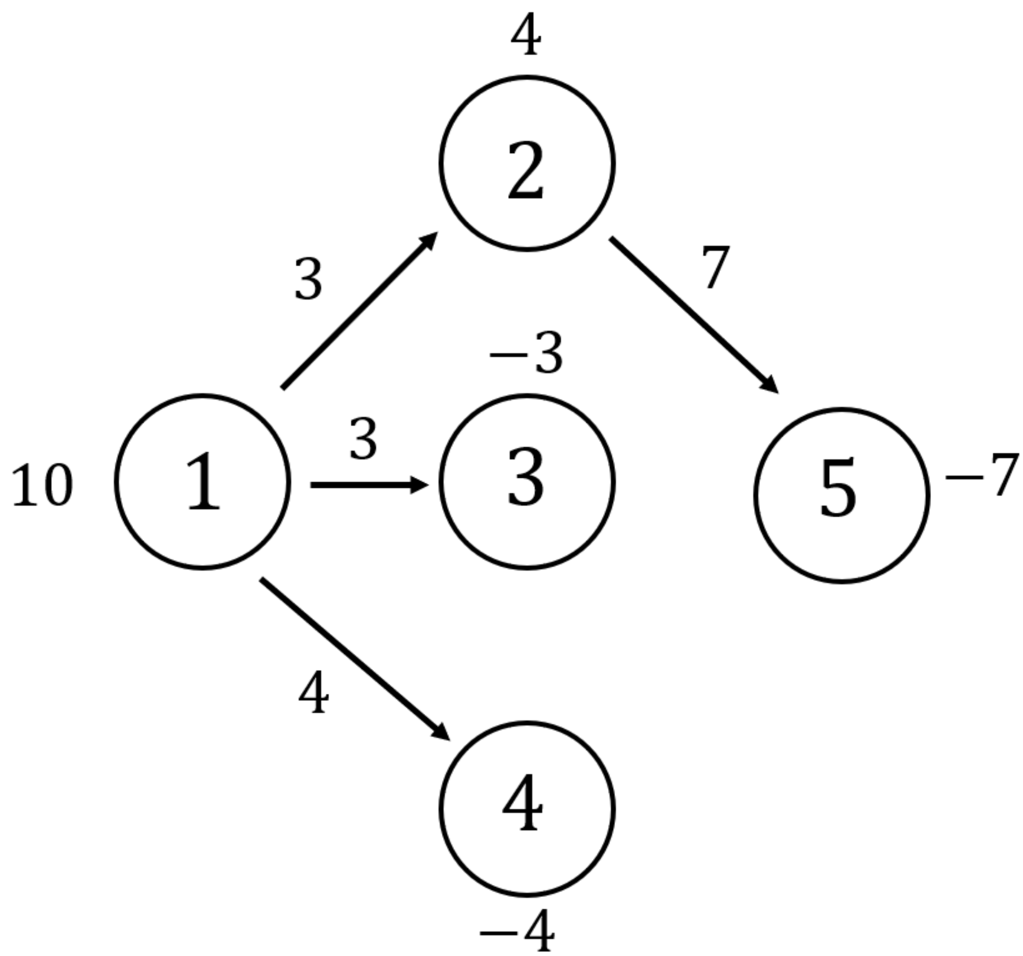


$$\overline{c_{14}} = 7 - 10 - 4 = -7 < 0 \text{ hence we add } 1 \to 4$$

STEP 2: Add $1 \to 4$



In that case, we increase the value of $1 \to 4$ by 4, until $3 \to 4$ disappears, and we get the following new tree



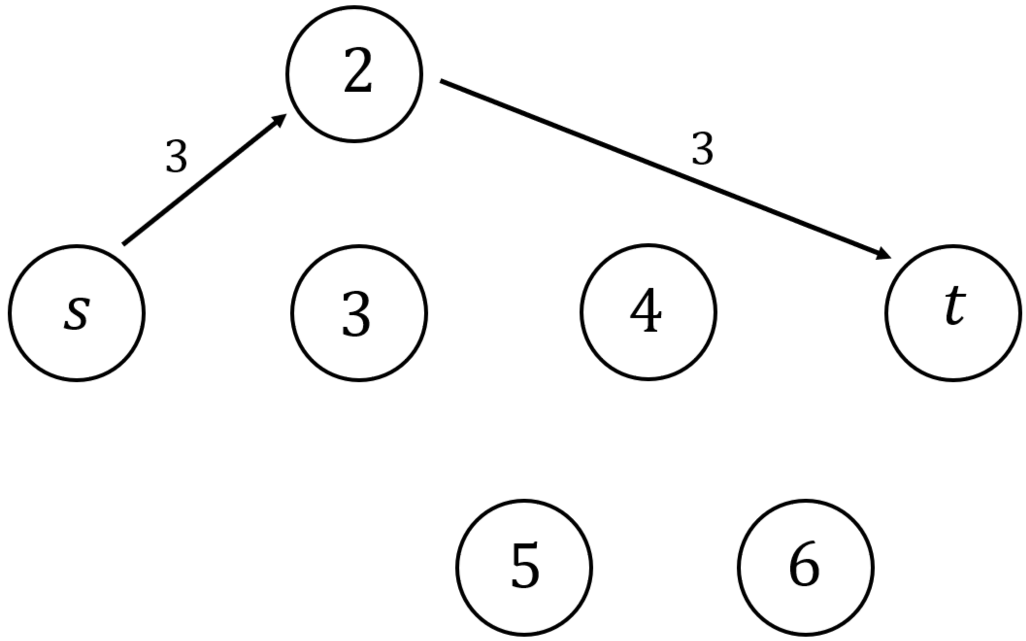
$$\overline{c_{34}} = 10 - 7 + 4 = 7 > 0$$
$$\overline{c_{45}} = 5 - 2 - 5 + 7 = 5 > 0$$
$$\overline{c_{35}} = 8 - 2 - 5 + 4 = 5 > 0$$

Since we've exhausted all the edges, the above tree is optimal

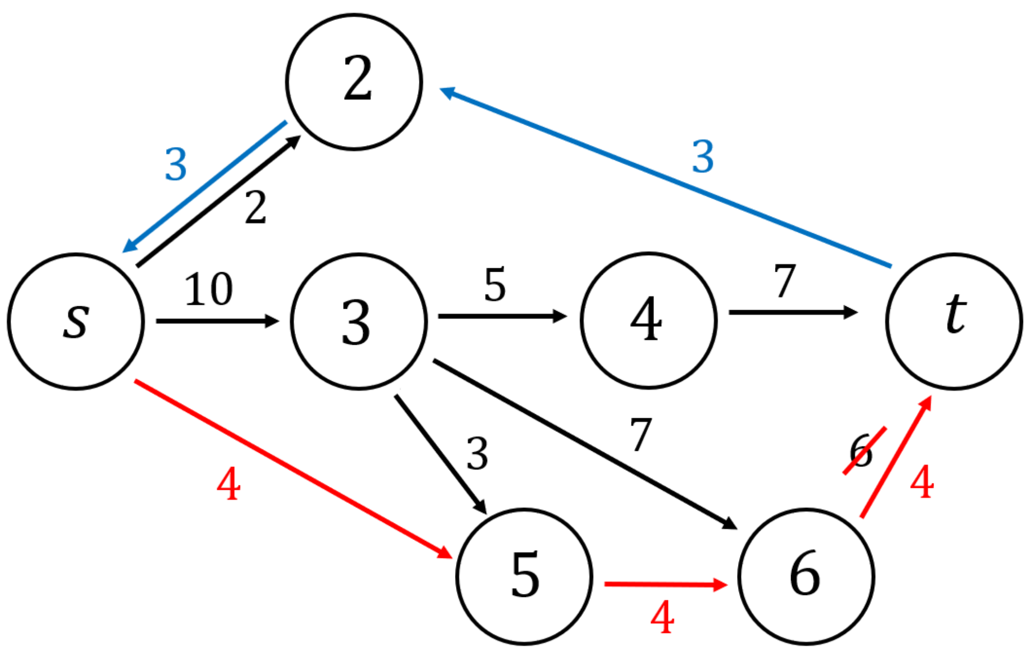
STEP 3: Optimal cost

$$(3 \times 5)+(3 \times 4)+(4 \times 7)+(7 \times 2) = 15+12+28+14 = 27+28+14 = 55+14 = 69$$

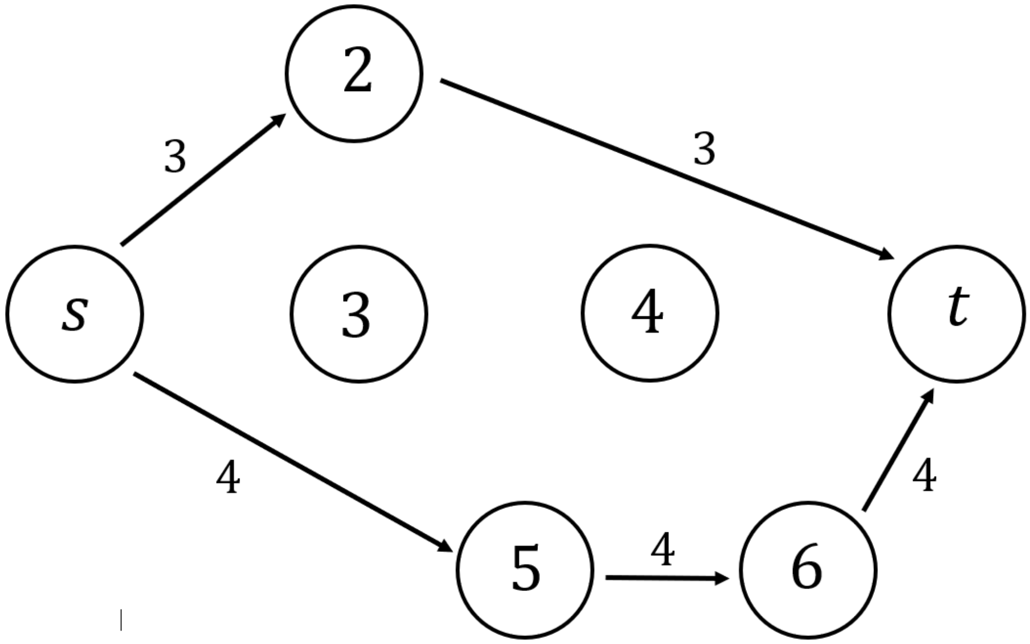
4. **STEP 1:** Find a path



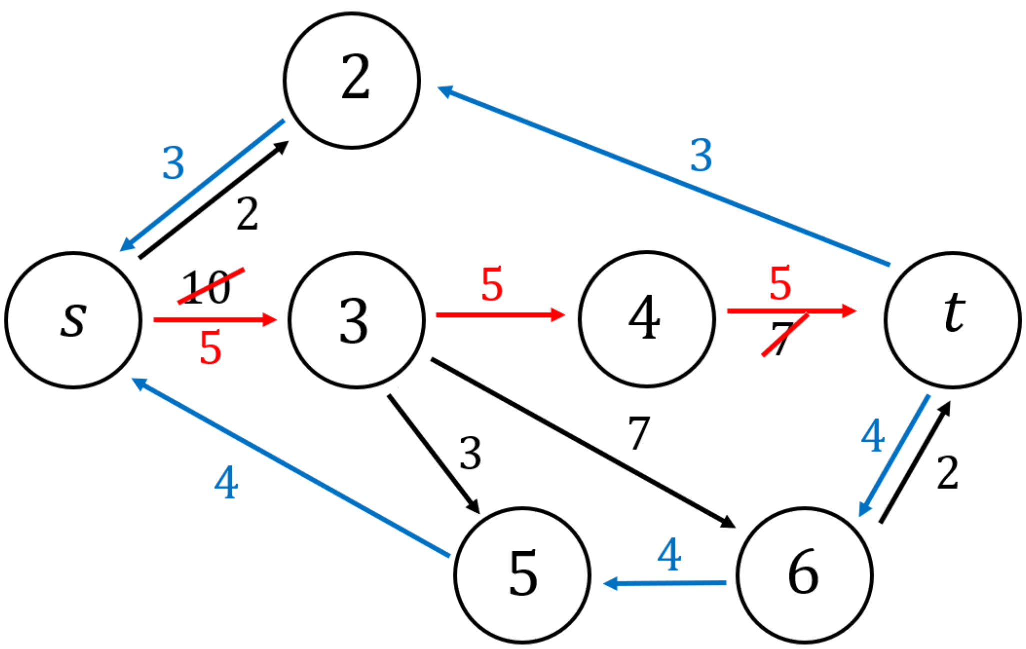
STEP 2: Residual graph and path



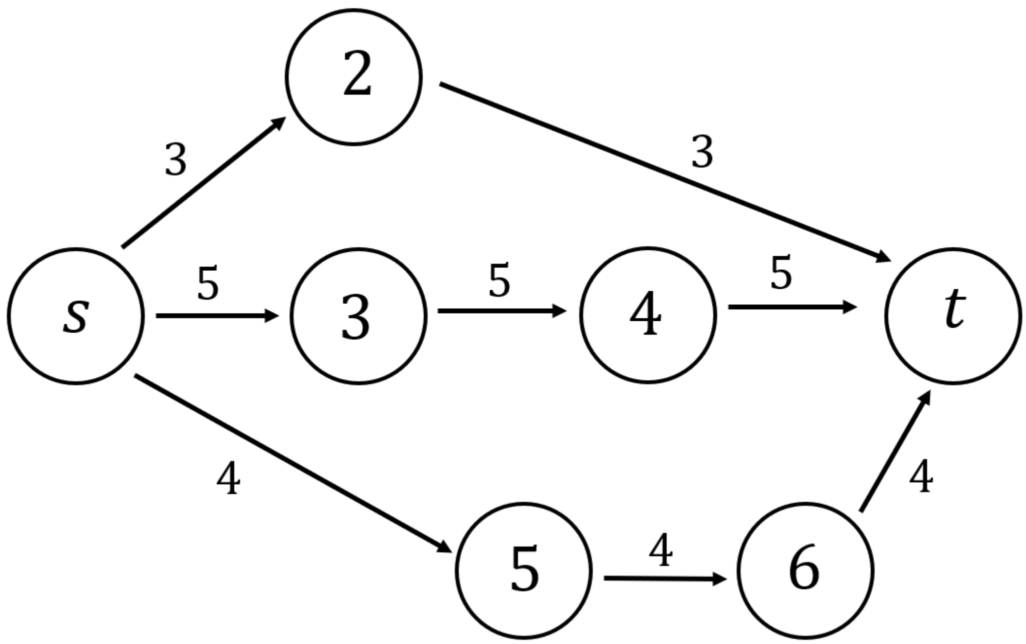
STEP 3: Add to previous path



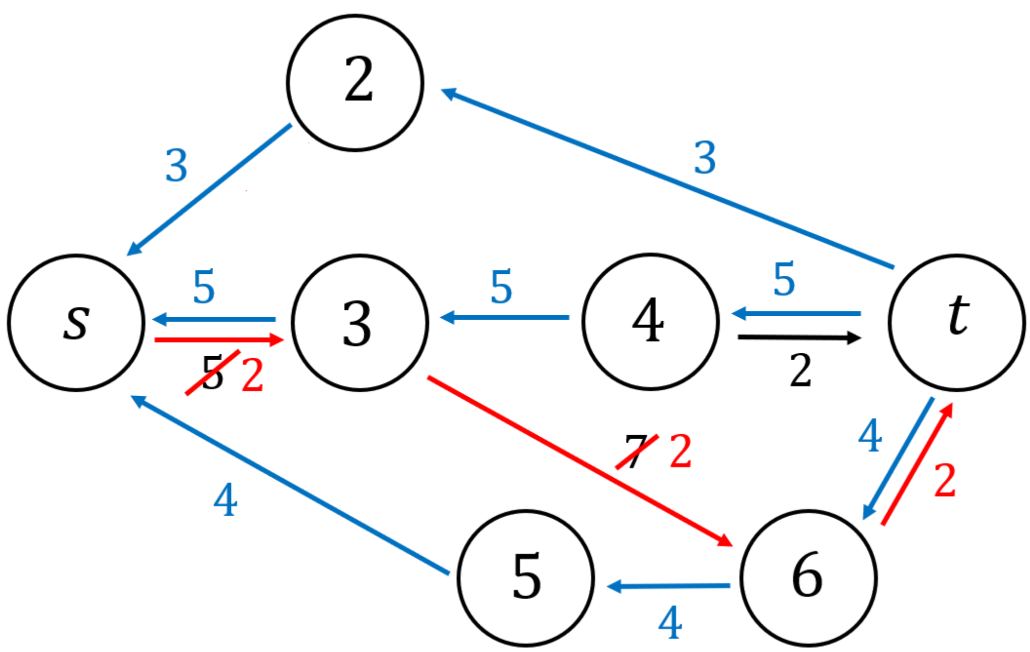
STEP 4: Residual graph and path



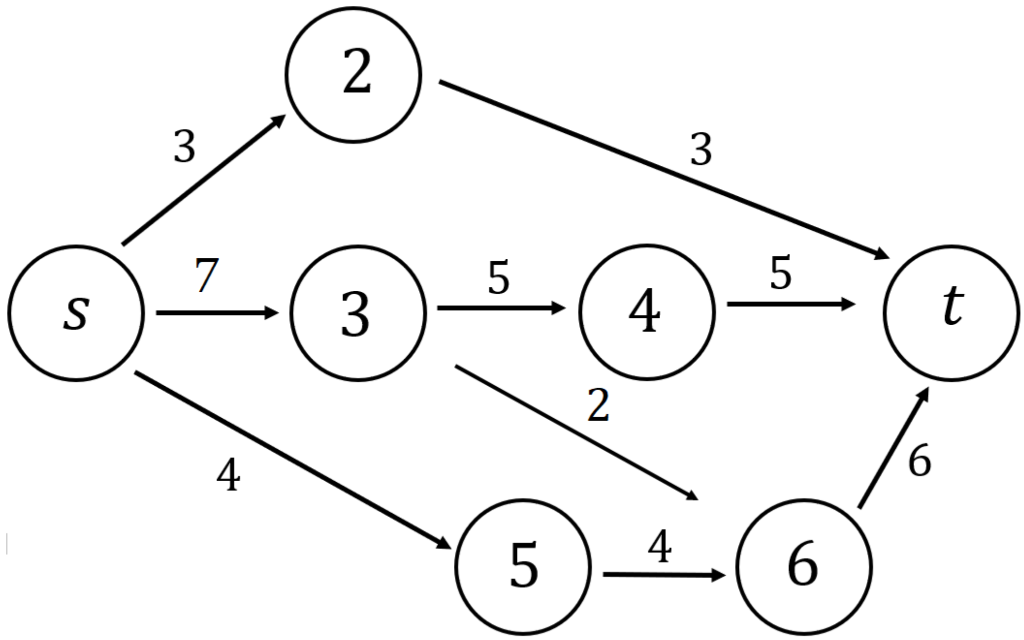
STEP 5: Add to previous path



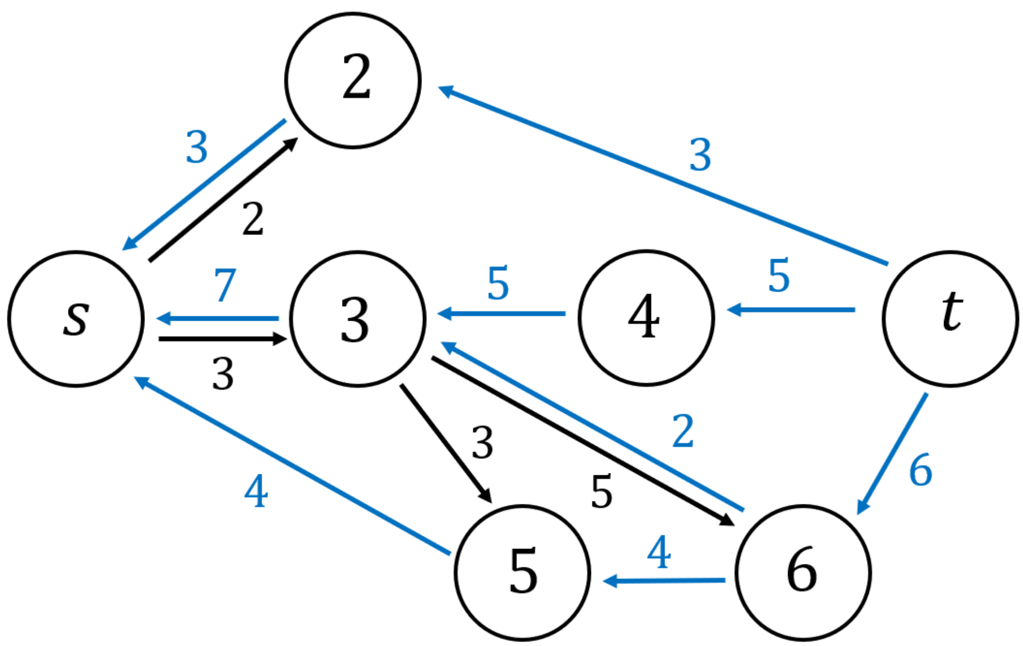
STEP 6: Residual graph and path



STEP 7: Add to previous path



STEP 8: Residual graph



There are no more paths, so the graph above is optimal.

Max flow: $3 + 7 + 4 = 14$

5. **Decision Variables:** x_i and y_i $(i = 1, 2, 3, 4)$

$$x_i = \begin{cases} 1 & \text{if you invest in company } i \\ 0 & \text{if not} \end{cases}$$

$$y_i = \text{number of shares bought from company } i$$

Objective Function:

$$z = 16y_1 + 22y_2 + 12y_3 + 8y_4 - (50x_1 + 70x_2 + 40x_3 + 30x_4)$$

Constraints:

$x_i \leq y_i$ (if you don't buy shares from the company, you don't invest in it)

(1) $x_1 + x_2 + x_3 + x_4 = 3$

(2) $x_2 \leq x_1$

(3) $x_2 + x_4 \leq 1$

$y_1 + y_2 + y_3 + y_4 \leq 100$ (at most 100 shares)

$50x_1 + 70x_2 + 40x_3 + 30x_4 \leq 140$ (initiation fee constraint)

IP Problem:

$$\begin{aligned} \max\ z = & (16y_1 + 22y_2 + 12y_3 + 8y_4) - (50x_1 + 70x_2 + 40x_3 + 30x_4) \\ \text{subject to}\ & x_i \leq y_i \quad (i = 1, 2, 3, 4) \\ & x_1 + x_2 + x_3 + x_4 = 3 \\ & x_2 \leq x_1 \\ & x_2 + x_4 \leq 1 \\ & y_1 + y_2 + y_3 + y_4 \leq 100 \\ & 50x_1 + 70x_2 + 40x_3 + 30x_4 \leq 140 \\ & x_i \in \{0, 1\}\ (i = 1, 2, 3, 4) \\ & y_i \geq 0 \text{ integer}\ \ (i = 1, 2, 3, 4) \end{aligned}$$

6.

$$f(x_1, x_2) = (x_1)^2 + 3(x_2)^2 - (x_1)(x_2) - 3(x_1) - 4(x_2) + 8$$

STEP 1: Critical Points

$$\begin{cases} f_{x_1} = 2x_1 - x_2 - 3 = 0 \\ f_{x_2} = 6x_2 - x_1 - 4 = 0 \end{cases}$$

Therefore we have to solve the system

$$\begin{cases} 2x_1 - x_2 = 3 \\ 6x_2 - x_1 = 4 \end{cases}$$

The first equation tells us $x_2 = 2x_1 - 3$ and plugging this into the second system we get

$$\begin{aligned} 6(2x_1 - 3) - x_1 &= 4 \\ 12x_1 - 18 - x_1 &= 4 \\ 11x_1 &= 22 \\ x_1 &= 2 \end{aligned}$$

$$x_2 = 2x_1 = 3 = 2(2) - 3 = 1$$

Critical Point: $(2, 1)$

STEP 2: $D^2 f$

$$D^2 f(x_1, x_2) = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 6 \end{bmatrix} = A$$

To show that f has a global min at $(2, 1)$, it's enough to show that f is convex

STEP 3(a): Method of Eigenvalues

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & -1 \\ -1 & 6-\lambda \end{vmatrix} \\ &= (2-\lambda)(6-\lambda) - (-1)(-1) \\ &= 12 - 2\lambda - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 8\lambda + 11 \\ &= (\lambda - 4)^2 - 16 + 11 \\ &= (\lambda - 4)^2 - 5 = 0 \end{aligned}$$

Which gives $\lambda = 4 \pm \sqrt{5}$

$\lambda = 4 + \sqrt{5} > 0$ and $\lambda = 4 - \sqrt{5} > 0$ (since $4 = \sqrt{16} > \sqrt{5}$)

Hence for all (x_1, x_2), $D^2 f(x_1, x_2)$ has positive eigenvalues, so f is convex.

STEP 3(b): Method of leading principal minors

$$D_2 = \begin{vmatrix} 2 & -1 \\ -1 & 6 \end{vmatrix} = (2)(6) - (-1)(-1) = 12 - 1 = 11 > 0$$

$$D_1 = \det[2] = 2 > 0$$

Both leading principal minors at (x_1, x_2) are positive

Hence f is convex

STEP 4: Answer: z has a global minimum at $(2, 1)$ and

$$\min z = z(2,1) = 2^2 + 3(1^2) - (2)(1) - 3(2) - 4(1) + 8 = \not{4} + 3 - 2 - 6 - \not{4} + 8 = 3$$

7. **STEP 1:** Following the hint, notice

$$a = \left(\frac{a}{a+b}\right)(a+b) + \left(\frac{b}{a+b}\right)0$$

Applying f to this identity, we get

$$f(a) = f\left(\left(\frac{a}{a+b}\right)(a+b) + \left(\frac{b}{a+b}\right)0\right)$$

But if $\lambda = \frac{a}{a+b}$ then $1-\lambda = 1 - \left(\frac{a}{a+b}\right) = \left(\frac{a+b}{a+b}\right) - \frac{a}{a+b} = \frac{b}{a+b}$ so

$$f(a) = f\left(\lambda(a+b) + (1-\lambda)0\right) \overset{C}{\leq} \lambda f(a+b) + (1-\lambda)\underbrace{f(0)}_{\leq 0} \leq \lambda f(a+b) = \left(\frac{a}{a+b}\right) f(a+b)$$

We used convexity in the $\overset{C}{\leq}$ step

STEP 2: Similarly, by writing

$$b = \left(\frac{b}{a+b}\right)(a+b) + \left(\frac{a}{a+b}\right)0$$

$$\text{We get: } f(b) \leq \left(\frac{b}{a+b}\right) f(a+b)$$

STEP 3: Adding up the two identities we get

$$f(a)+f(b) \leq \left(\frac{a}{a+b}\right) f(a+b) + \left(\frac{b}{a+b}\right) f(a+b) = \left(\frac{a+b}{a+b}\right) f(a+b) = f(a+b) \checkmark$$