LECTURE: VARIATION OF PARAMETERS

Welcome to the **second** method for solving inhomogeneous equations: Var of Par. It's harder to use **but** it requires no guessing whatsoever. For this, we need two preliminaries:

1. Cramer's Rule

Video: Cramer's Rule

The first ingredient is Cramer's Rule, which is a direct way of solving systems of equations:

Definition:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1:

$$\text{Solve } \begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

For x, you take the determinant of your matrix, but replace the *first* column by the right-hand-side

$$x = \frac{\begin{vmatrix} 9 & 3 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{(9)(-1) - 3(-5)}{(-5)(-1) - (3)(3)} = \frac{6}{-4} = -\frac{3}{2}$$

For y, you take the determinant of your matrix, but replace the *second* column by the right-hand-side

$$y = \frac{\begin{vmatrix} -5 & 9 \\ 3 & -5 \end{vmatrix}}{\begin{vmatrix} -5 & 3 \\ 3 & -1 \end{vmatrix}} = \frac{(-5)(-5) - (9)(3)}{-4} = \frac{25 - 27}{-4} = \frac{-2}{-4} = \frac{1}{2}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$$

Note: In case you're interested, here's a proof:

Video: Cramer Rule Proof

2. The Wronskian

The second ingredient that we'll need is the Wronskian (matrix)

Definition:

The Wronskian matrix of two functions $f(t)$ and $g(t)$ is

$$\begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix}$$

Example 2:

The Wronskian matrix of t^2 and t^3 is

$$\begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix}$$

You just put f and g on your first row, and differentiate that row.

Note: The determinant of this is called the Wronskian, and is a powerful tool to solve linear ODE, see separate handout.

3. Variation of Parameters

Video: Variation of Parameters

Example 3: (Model Problem)

Use **Variation of Parameters** to solve

$$y'' + y = \sec(t)$$

STEP 1: Put into standard form (coefficient of y'' is 1) ✓

STEP 2: Homogeneous

$$y'' + y = 0 \Rightarrow y_0 = A\cos(t) + B\sin(t)$$

STEP 3: Particular

Main Idea: Guess y_p to be of the same form, that is

$$y_p = u(t)\cos(t) + v(t)\sin(t)$$

Where $u(t)$ and $v(t)$ two unknown **functions**

If you plug this into the ODE you eventually get the system

$$\begin{cases} \cos(t)u'(t) + \sin(t)v'(t) & = 0 \\ (-\sin(t))\,u'(t) + (\cos(t))\,v'(t) & = \sec(t) \end{cases}$$

(We'll derive this below)

Which you can rewrite as:

Var of Par equations:

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(t) \end{bmatrix}$$

The left hand side is *precisely* the Wronskian (matrix) ☺

$$\textbf{Mnemonic:} \quad \left[\text{ Wronskian Matrix } \right] \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \text{Inhomogeneous} \end{bmatrix}$$

STEP 4: Solve using Cramer's rule

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(t) \end{bmatrix}$$

$$\begin{aligned} u'(t) = & \frac{\begin{vmatrix} 0 & \sin(t) \\ \sec(t) & \cos(t) \end{vmatrix}}{\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}} = \frac{0\cos(t) - \sin(t)\sec(t)}{\cos(t)\cos(t) - \sin(t)\left(-\sin(t)\right)} = \frac{-\sin(t)\sec(t)}{1} \\ & = -\sin(t)\left(\frac{1}{\cos(t)}\right) = -\tan(t) \end{aligned}$$

$$v'(t) = \frac{\begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \sec(t) \end{vmatrix}}{1} = \cos(t)\sec(t) - 0 = \cos(t)\left(\frac{1}{\cos(t)}\right) = 1$$

STEP 5: Integrate

$$u'(t) = -\tan(t) \Rightarrow u(t) = \int -\tan(t)dt = -\ln|\sec(t)|$$

(No constants since we only want one particular solution)

$$v'(t) = 1 \Rightarrow v(t) = \int 1dt = t$$

$$y_p = u(t)\cos(t) + v(t)\sin(t) = -(\ln|\sec(t)|)\cos(t) + t\sin(t)$$

STEP 6: Answer

$$y(t) = y_0 + y_p = A\cos(t) + B\sin(t) - (\ln|\sec(t)|)\cos(t) + t\sin(t)$$

4. Why this works

Here is how to obtain the var of par formulas

STEP 1: As before, the hom solution is $y_0 = A\cos(t) + B\sin(t)$

And the idea is to guess

$$y_p = u(t)\cos(t) + v(t)\sin(t)$$

STEP 2: First Equation

$$\begin{aligned}(y_p)' =& u'(t)\cos(t) - u(t)\sin(t) + v'(t)\sin(t) + v(t)\cos(t)\\ =& \cos(t)u'(t) + \sin(t)v'(t) - u(t)\sin(t) + v(t)\cos(t)\end{aligned}$$

Since we're just looking for *one* particular solution, it's ok to simplify our work and assume that the term in blue is 0.

Note: There's no reason why we *have* to set it equal to 0, and it's possible that we don't get any solutions this way, but luckily we do! If you don't do this, you will get some nasty $u''(t)$ and $v''(t)$ terms

This gives us our first equation:

$$\cos(t)u'(t) + \sin(t)v'(t) = 0$$

And $(y_p)'$ then becomes

$$(y_p)' = -u(t)\sin(t) + v(t)\cos(t)$$

STEP 3: Second Equation

$$\begin{aligned}(y_p)'' &= (-u(t)\sin(t) + v(t)\cos(t))' \\ &= -u'(t)\sin(t) - u(t)\cos(t) + v'(t)\cos(t) - v(t)\sin(t)\end{aligned}$$

$$\begin{aligned}&(y_p)'' + y_p \\ =& -u'(t)\sin(t) - u(t)\cos(t) + v'(t)\cos(t) - v(t)\sin(t) + (u(t)\cos(t) + v(t)\sin(t)) \\ =& -u'(t)\sin(t) + \cos(t)v'(t) + u(t)\left(-\cancel{\cos(t)} + \cancel{\cos(t)}\right) + v(t)\left(-\cancel{\sin(t)} + \cancel{\sin(t)}\right) \\ =& -u'(t)\sin(t) + \cos(t)v'(t)\end{aligned}$$

Although not apparent at first sight, the cancellation in the last step actually follows because cos and sin are homogeneous solutions

STEP 4:

Therefore $(y_p)'' + (y_p) = \sec(t)$ just becomes

$$u'(t)\left(-\sin(t)\right) + v'(t)\left(\cos(t)\right) = \sec(t)$$

Which is our second equation!

STEP 5: To summarize, our equations are

$$\begin{cases} \cos(t)u'(t) + \sin(t)v'(t) = 0 \\ (-\sin(t))\, u'(t) + (\cos(t))\, v'(t) = \sec(t) \end{cases}$$

$$\text{Which becomes } \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \sec(t) \end{bmatrix}$$

Which are our var of par equations!

5. Variable Coefficients

The cool thing is that this method allows us to solve non-constant coefficient ODE, provided we know the homogeneous solution

Example 4:

$$t^2\, y'' - 2y = 3t^2 - 1$$

Assume t^2 and $\frac{1}{t}$ solve the homogeneous equation (will be given)

STEP 1: Standard Form

$$y'' - \left(\frac{2}{t^2}\right) y = \frac{3t^2 - 1}{t^2} = 3 - \frac{1}{t^2}$$

STEP 2: Homogeneous Solution

$$y_0 = At^2 + B\left(\frac{1}{t}\right)$$

STEP 3: Variation of Parameters

$$y_p = u(t)t^2 + v(t)\left(\frac{1}{t}\right)$$

$$\begin{bmatrix} t^2 & \frac{1}{t} \\ 2t & -\frac{1}{t^2} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 - \frac{1}{t^2} \end{bmatrix}$$

STEP 4: Cramer's Rule

$$u'(t) = \frac{\begin{vmatrix} 0 & \frac{1}{t} \\ 3 - \frac{1}{t^2} & -\frac{1}{t^2} \end{vmatrix}}{\begin{vmatrix} t^2 & \frac{1}{t} \\ 2t & -\frac{1}{t^2} \end{vmatrix}} = \frac{0 - \frac{1}{t}\left(3 - \frac{1}{t^2}\right)}{-1-2} = \frac{-\frac{3}{t} + \frac{1}{t^3}}{-3} = \frac{1}{t} - \frac{1}{3t^3}$$

$$v'(t) = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & 3 - \frac{1}{t^2} \end{vmatrix}}{-3} = \frac{t^2\left(3 - \frac{1}{t^2}\right) - 0}{-3} = \frac{3t^2 - 1}{-3} = -t^2 + \frac{1}{3}$$

STEP 5: Integrate

$$u'(t) = \frac{1}{t} - \frac{1}{3}t^{-3} \Rightarrow u(t) = \ln|t| - \frac{1}{3}\left(-\frac{1}{2}t^{-2}\right) = \ln|t| + \frac{1}{6t^2}$$

$$v'(t) = -t^2 + \frac{1}{3} \Rightarrow v(t) = -\frac{t^3}{3} + \frac{t}{3}$$

STEP 6:

$$\begin{aligned} y_p &= u(t)t^2 + v(t)\frac{1}{t} \\ &= \left(\ln|t| + \frac{1}{6t^2}\right)t^2 + \left(-\frac{t^3}{3} + \frac{t}{3}\right)\frac{1}{t} \\ &= t^2\ln|t| + \frac{1}{6} - \frac{t^2}{3} + \frac{1}{3} \\ y_p &= t^2\ln|t| - \frac{t^2}{3} + \frac{1}{2} \end{aligned}$$

STEP 7: General Solution

$$y = At^2 + B\left(\frac{1}{t}\right) + t^2\ln|t| - \frac{t^2}{3} + \frac{1}{2}$$

$$y = At^2 + B\left(\frac{1}{t}\right) + t^2\ln|t| + \frac{1}{2} \qquad \text{(Since } A \text{ is arbitrary)}$$

6. Extra practice

Example 5: (extra practice)

Use **Variation of Parameters** to solve

$$y'' - 5y' + 6y = e^t$$

STEP 1: Standard Form ✓

STEP 2: Homogeneous

$$y_0 = Ae^{2t} + Be^{3t}$$

STEP 3: Particular

$$y_p = u(t)e^{2t} + v(t)e^{3t}$$

$$\begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

STEP 4:

$$\begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

$$u'(t) = \frac{\begin{vmatrix} 0 & e^{3t} \\ e^{t} & 3e^{3t} \end{vmatrix}}{\begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix}} = \frac{0 - e^{3t}e^{t}}{\left(3e^{3t}\right)e^{2t} - e^{3t}\left(2e^{2t}\right)} = \frac{-e^{4t}}{e^{5t}} = -e^{-t}$$

$$v'(t) = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & e^{t} \end{vmatrix}}{e^{5t}} = \frac{e^{2t}e^{t} - 0}{e^{5t}} = \frac{e^{3t}}{e^{5t}} = e^{-2t}$$

STEP 5: Integrate

$$u'(t) = -e^{-t} \Rightarrow u(t) = \int -e^{-t}dt = e^{-t}$$

$$v'(t) = e^{-2t} \Rightarrow v(t) = \int e^{-2t} = -\frac{1}{2}e^{-2t}$$

$$y_p(t) = u(t)e^{2t} + v(t)e^{3t} = e^{-t}e^{2t} - \frac{1}{2}e^{-2t}e^{3t} = e^{t} - \frac{1}{2}e^{t} = \frac{1}{2}e^{t}$$

STEP 6: Answer

$$y(t) = Ae^{2t} + Be^{3t} + \frac{1}{2}e^{t}$$

Remark: Even though undetermined coefficients here is easier, this method works **every** time and doesn't require any guessing.

Example 6:

$$y'' + y = \tan(t)$$

STEP 1: Standard form ✓

STEP 2: Homogeneous Solution

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

$$y_0 = A\cos(t) + B\sin(t)$$

STEP 3: Variation of Parameters

$$y_p = u(t)\cos(t) + v(t)\sin(t)$$

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(t) \end{bmatrix}$$

STEP 4: Cramer's Rule

$$u'(t) = \frac{\begin{vmatrix} 0 & \sin(t) \\ \tan(t) & \cos(t) \end{vmatrix}}{\begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix}} = \frac{0 - \sin(t)\tan(t)}{\cos^2(t) + \sin^2(t)} = -\sin(t)\tan(t)$$

$$v'(t) = \frac{\begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \tan(t) \end{vmatrix}}{1} = \tan(t)\cos(t) = \frac{\sin(t)}{\cos(t)}\cos(t) = \sin(t)$$

STEP 5: Integrate

$$u'(t) = -\sin(t)\tan(t) \Rightarrow u(t) = \int -\sin(t)\tan(t)dt \;\; \text{Sorry} \;\; ☹$$

$$\begin{aligned}
\int -\sin(t)\tan(t) &= \int -\sin(t)\left(\frac{\sin(t)}{\cos(t)}\right)dt \\
&= \int -\frac{\sin^2(t)}{\cos(t)}dt \\
&= \int -\left(\frac{1-\cos^2(t)}{\cos(t)}\right)dt \\
&= \int -\frac{1}{\cos(t)} + \cos(t)dt \\
&= \int -\sec(t) + \cos(t)dt \\
u(t) &= -\ln|\sec(t)+\tan(t)| + \sin(t)
\end{aligned}$$

$$v'(t) = \sin(t) \Rightarrow v(t) = \int \sin(t) = -\cos(t)$$

STEP 6:

$$\begin{aligned}
y_p =& u(t)\cos(t) + v(t)\sin(t) \\
=& (-\ln|\sec(t)+\tan(t)| + \sin(t))\cos(t) + (-\cos(t)\sin(t)) \\
=& -\ln|\sec(t)+\tan(t)|\cos(t) + \cancel{\sin(t)\cos(t)} - \cancel{\cos(t)\sin(t)} \\
y_p =& -\ln|\sec(t)+\tan(t)|\cos(t)
\end{aligned}$$

$$y = y_0 + y_p = A\cos(t) + B\sin(t) - \ln|\sec(t)+\tan(t)|\cos(t)$$