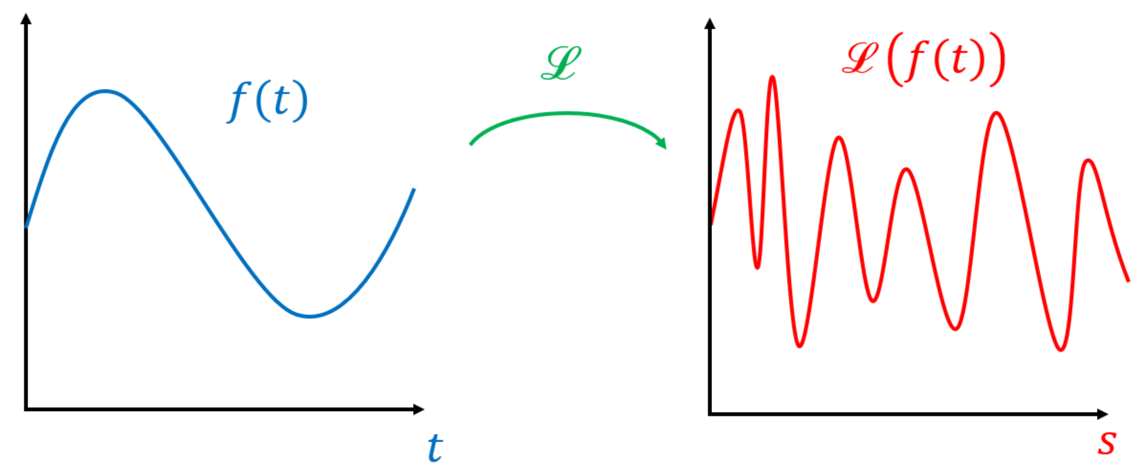
LECTURE: THE LAPLACE TRANSFORM

1. Introduction

Welcome to the Queen of Applied Math: the Laplace Transform.

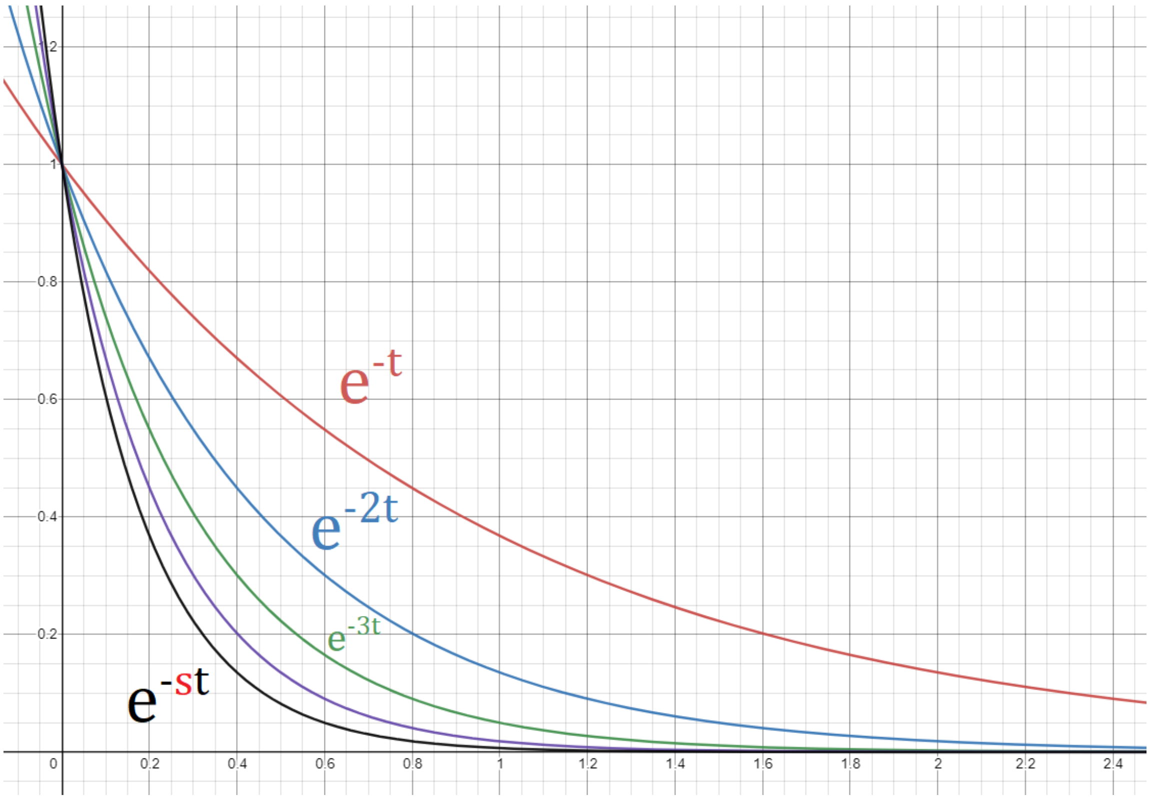
It transforms functions of time t into functions of frequency s



Definition:

$$\mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}dt$$

Intuitively: For every s, $\mathcal{L}\left\{f\right\}$ gives you a weighted average of f with weight e^{-st} (which becomes smaller and smaller)



Demo: e^{-st} for various values of s

2. Examples

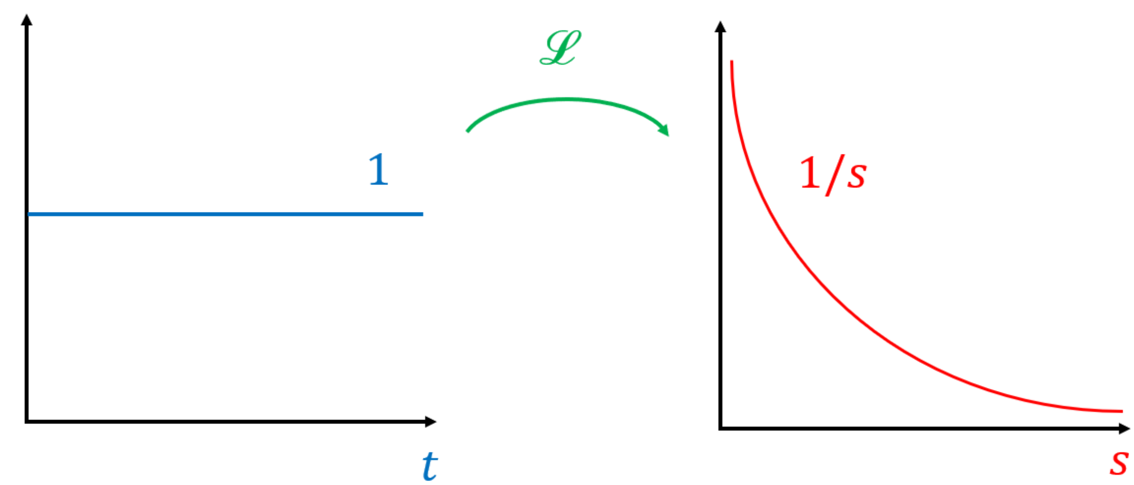
Video: Laplace Transform Marathon (courtesy blackpenredpen)

Example 1:

$$\mathcal{L}\{1\}$$

$$\mathcal{L}\left\{1\right\} = \int_0^\infty 1e^{-st}dt = \int_0^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty}$$
$$= \left(\lim_{t\to\infty}\frac{e^{-st}}{-s}\right) - \frac{e^{-s(0)}}{-s} = 0 - \left(\frac{1}{-s}\right) = \frac{1}{s}$$

$$\mathcal{L} \text{ transforms 1 into } \frac{1}{s}$$



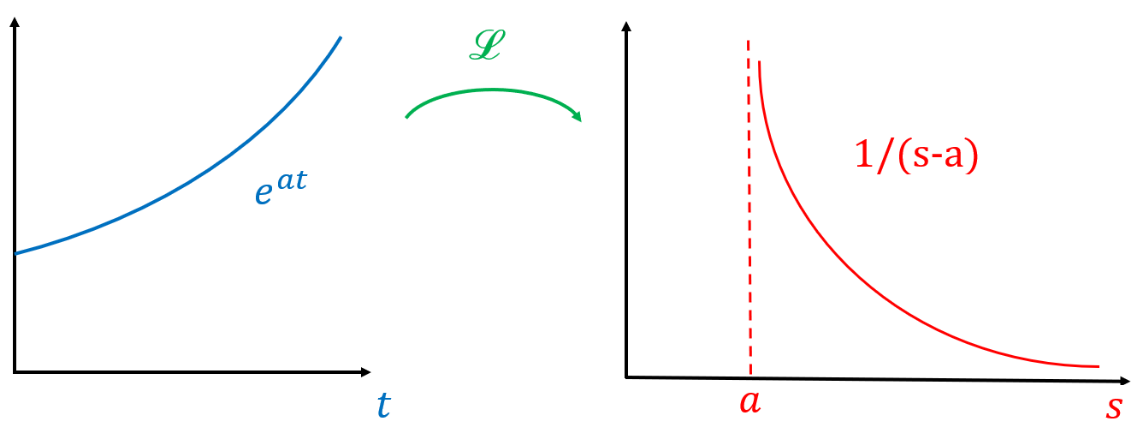
Example 2:

$$\mathcal{L}\left\{e^{2t}\right\}$$

$$
\begin{aligned}
\mathcal{L}\left\{e^{2t}\right\} &= \int_0^\infty e^{2t}e^{-st}dt \\
&= \int_0^\infty e^{(2-s)t}dt \\
&= \left[\frac{e^{(2-s)t}}{2-s}\right]_{t=0}^{t=\infty} \\
&= \left(\lim_{t\to\infty}\frac{e^{(2-s)t}}{2-s}\right) - \frac{e^{(2-s)(0)}}{2-s} \\
&=0 - \frac{1}{2-s} \qquad \text{(Assume } s > 2\text{)} \\
&=\frac{1}{s-2}
\end{aligned}
$$

Fact:

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \qquad s > a$$



Example 3:

$$\mathcal{L}\left\{e^{-3t}\right\} = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

Example 4:

$$\mathcal{L}\left\{\cos(2t)\right\} \text{ and } \mathcal{L}\left\{\sin(2t)\right\}$$

$$\mathcal{L}\left\{\cos(2t)\right\} = \int_0^\infty \cos(2t)e^{-st}dt$$

Either integrate by parts twice (boo!) or:

Cool Peyam Trick: Consider

$$\mathcal{L}\left\{e^{(2t)i}\right\}$$

STEP 1: On the one hand, by definition of $e^{(2t)i}$

$$\mathcal{L}\left\{e^{(2t)i}\right\} = \mathcal{L}\left\{\cos(2t) + i\sin(2t)\right\} = \mathcal{L}\left\{\cos(2t)\right\} + i\mathcal{L}\left\{\sin(2t)\right\}$$

STEP 2: On the other hand, using $\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$ with $a = 2i$ we get

$$\begin{aligned}\mathcal{L}\left\{e^{2ti}\right\} =& \mathcal{L}\left\{e^{(2i)t}\right\} = \frac{1}{s-2i} = \left(\frac{1}{s-2i}\right)\left(\frac{s+2i}{s+2i}\right) \\ =& \frac{s+2i}{(s-2i)(s+2i)} = \frac{s+2i}{s^2-(2i)^2} = \frac{s+2i}{s^2+4}\end{aligned}$$

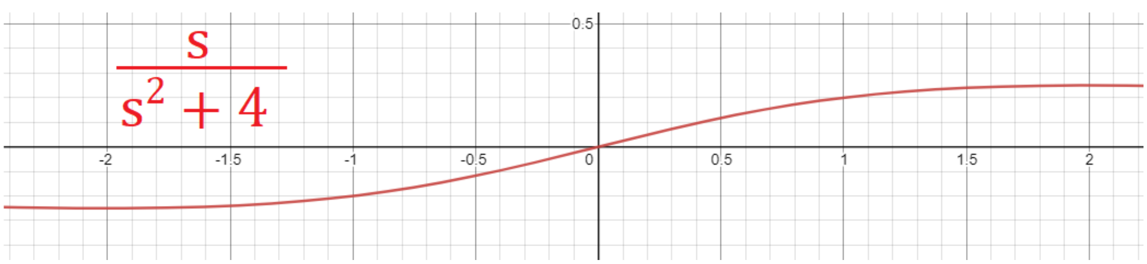
$$\mathcal{L}\left\{\cos(2t)\right\} + i\mathcal{L}\left\{\sin(2t)\right\} = \frac{s}{s^2+4} + \left(\frac{2}{s^2+4}\right)i$$

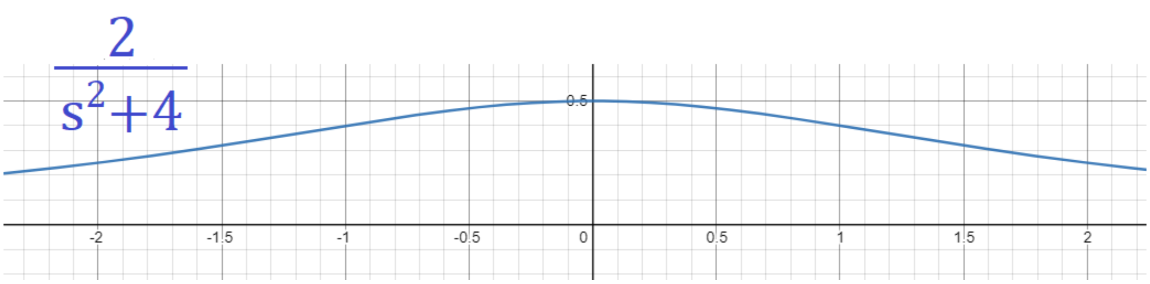
STEP 3: Comparing the real and imaginary parts, we get

$$\mathcal{L}\left\{\cos(2t)\right\} = \frac{s}{s^2+4}$$
$$\mathcal{L}\left\{\sin(2t)\right\} = \frac{2}{s^2+4}$$

Fact:

$$\mathcal{L}\left\{\cos(at)\right\} = \frac{s}{s^2+a^2}$$
$$\mathcal{L}\left\{\sin(at)\right\} = \frac{a}{s^2+a^2}$$





Example 5:

$$\mathcal{L}\left\{2\sin(3t) + 4e^{5t}\right\}$$

The Laplace Transform is linear, so this is just:

$$2\mathcal{L}\left\{\sin(3t)\right\} + 4\mathcal{L}\left\{e^{5t}\right\} = 2\left(\frac{3}{s^2+9}\right) + 4\left(\frac{1}{s-5}\right) = \frac{6}{s^2+9} + \frac{4}{s-5}$$

Finally it's useful sometimes do this in reverse:

Example 6:

Which function has Laplace transform

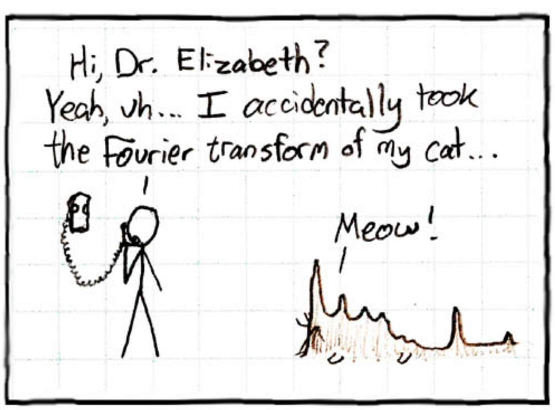
$$\frac{2}{s+1} - \frac{4}{s-3}$$

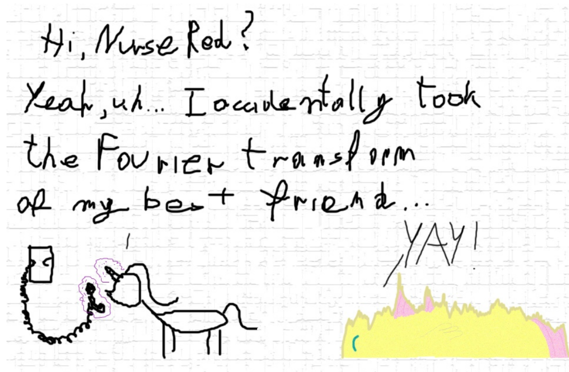
Note: This is sometimes written as

"Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{4}{s-3}\right\}$"

Recalling the rule $\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$ we get

$$2e^{-t} - 4e^{3t}$$





3. Tabular Integration

Video: Tabular Integration

Here's a really cool trick for Laplace transforms of power functions

Example 7:

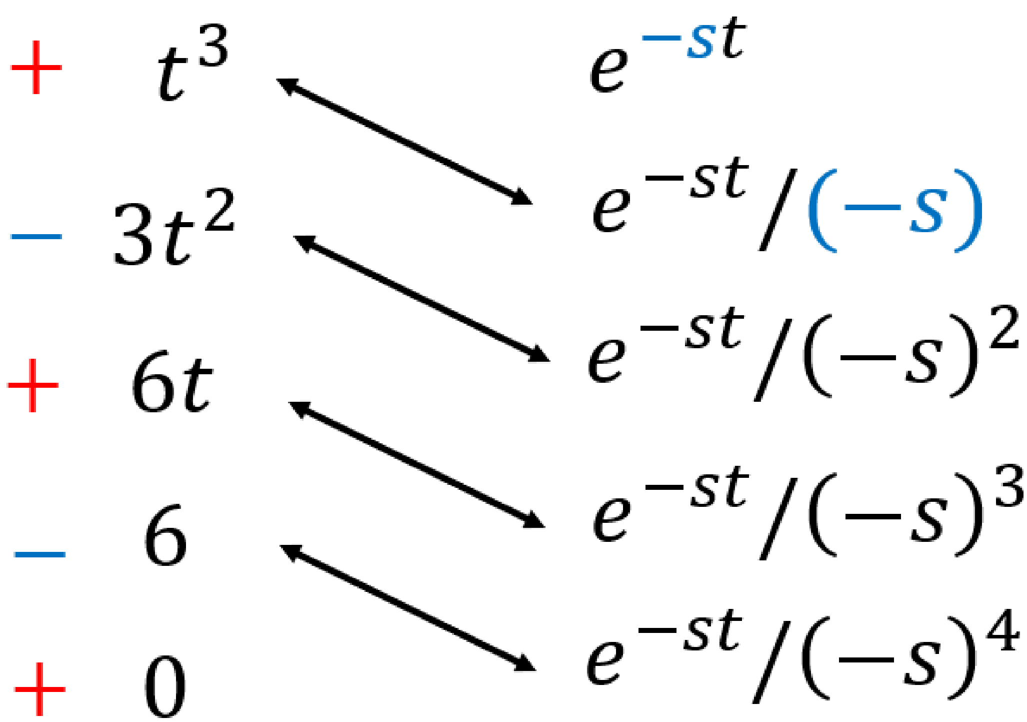
$$\mathcal{L}\left\{t^3\right\}$$

$$\mathcal{L}\left\{t^3\right\} = \int_0^\infty t^3 e^{-st} dt$$

STEP 1: Put t^3 on the left hand side and e^{-st} on the right hand side.

STEP 2: Differentiate t^3 until you get 0 and integrate e^{-st}

STEP 3: Then cross multiply and alternate signs



What you get is (assuming the terms at ∞ are 0)

$$\left[+t^3\left(\frac{e^{-st}}{-s}\right) - 3t^2\left(\frac{e^{-st}}{(-s)^2}\right) + 6t\left(\frac{e^{-st}}{(-s)^3}\right) - 6\left(\frac{e^{-st}}{(-s)^4}\right)\right]_{t=0}^{t=\infty}$$

$$= (0 - 0 + 0 - 0) - \left(0^3 - 3(0)^2 + 6(0) - 6\left(\frac{e^{-s(0)}}{(-s)^4}\right)\right)$$

$$= 6\left(\frac{1}{s^4}\right) = \frac{6}{s^4}$$

Fact:

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

4. Laplace Miracle

Laplace Miracle:

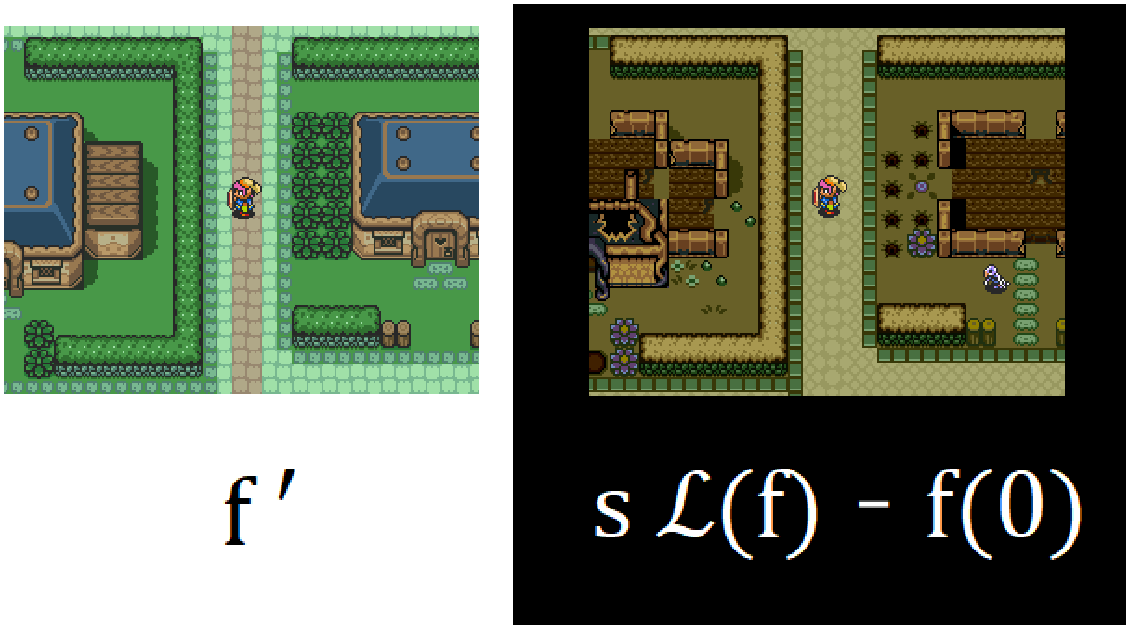
$$\mathcal{L}\left\{f'(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0)$$

In other words, the Laplace transform turns **differentiation** (hard) into **multiplication** (easy)

Why?

$$\begin{aligned}
\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty f'(t)e^{-st}dt \\
&\overset{\text{IBP}}{=} \left[f(t)e^{-st}\right]_{t=0}^{t=\infty} - \int_0^\infty f(t)\left(-se^{-st}\right)dt \\
&= 0 - f(0)e^{-s(0)} + s\int_0^\infty f(t)e^{-st}dt \\
&= -f(0) + s\mathcal{L}\left\{f(t)\right\}
\end{aligned}$$

Analogy: The Laplace transform turns the real world (with t) into a shadow world (with s). In this shadow world, differentiation becomes multiplication, which turns differential equations (hard) into algebra equations (easy)



Consequence:

$$\begin{aligned}\mathcal{L}\left\{f''(t)\right\} =& \mathcal{L}\left\{(f'(t))'\right\}\\ =& s\mathcal{L}\left\{f'(t)\right\} - f'(0)\\ =& s\left(s\mathcal{L}\left\{f(t)\right\} - f(0)\right) - f'(0)\\ =& s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)\end{aligned}$$

Consequence:

$$\mathcal{L}\left\{f''(t)\right\} = s^2\mathcal{L}\left\{f(t)\right\} - sf(0) - f'(0)$$

We'll use this next time to solve ODE