# LECTURE: SYSTEMS OF ODE (II)

**Today:** We're finally ready to solve systems of ODE!

## 1. Solving Systems of ODE

**Video:** Systems of ODE

**Example 1:**

$$\text{Solve } \mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$$

It just boils down to finding the eigenvalues/vectors of $A$!

**STEP 1: Eigenvalues**

$$\begin{aligned} |A - \lambda I| = & \begin{vmatrix} 7-\lambda & -3 \\ 10 & -4-\lambda \end{vmatrix} \\ = & (7-\lambda)(-4-\lambda) - (-3)(10) \\ = & -28 - 7\lambda + 4\lambda + \lambda^2 + 30 \\ = & \lambda^2 - 3\lambda + 2 \\ = & (\lambda - 1)(\lambda - 2) = 0 \end{aligned}$$

$$\lambda = 1 \text{ or } \lambda = 2$$

**STEP 2:** $\lambda = 1$

$$\begin{aligned}
\text{Nul }(A - 1I) &= \left[\begin{array}{cc|c} 7-1 & -3 & 0 \\ 10 & -4-1 & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} 6 & -3 & 0 \\ 10 & -5 & 0 \end{array}\right] \\
&\xrightarrow{(\div 3)R_1\ (\div 5)R_2} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

Hence $2x - y = 0$ so $y = 2x$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 1$

**STEP 3:** $\lambda = 2$

$$\begin{aligned}
\text{Nul }(A - 2I) &= \left[\begin{array}{cc|c} 7-2 & -3 & 0 \\ 10 & -4-2 & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} 5 & -3 & 0 \\ 10 & -6 & 0 \end{array}\right] \\
&\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 5 & -3 & 0 \\ 10-2(5) & -6-2(-3) & 0 \end{array}\right] \\
&\longrightarrow \left[\begin{array}{cc|c} 5 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]
\end{aligned}$$

Hence $5x - 3y = 0$

$x = 3$ and $y = 5$ work, so an eigenvector for $\lambda = 2$ is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

**STEP 4: Solution:**

$\lambda = 1 \rightsquigarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda = 2 \rightsquigarrow \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ hence the solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = C_1 e^{1t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Here $C_1$ and $C_2$ are constants.

The point is the eigenvectors go with the corresponding eigenvalues.

## 2. Why this works

Here is why this works!

**STEP 1: Original Problem:**

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$$

At this point we are stuck! As is usual in math, before solving a hard problem, let's solve an easier version first:

**STEP 2: Easier System:** Let's solve

$$(\star) \qquad \begin{cases} y_1'(t) = & 1y_1(t) \\ y_2'(t) = & 2y_2(t) \end{cases}$$

This is **much** easier because we can solve both equations separately:

$$\begin{cases} y_1(t) = & C_1 e^{t} \\ y_2(t) = & C_2 e^{2t} \end{cases}$$

Where $C_1$ and $C_2$ are constants.

**Important Observation:** ($\star$) can be written as

$$\mathbf{y}'(t) = D\mathbf{y}(t) \text{ where } \mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ is diagonal}$$

**Moral:** System with *diagonal* matrices are *easy* to solve

**STEP 3:** Back to $\mathbf{x}' = A\mathbf{x}$

Given the previous step, the idea is to turn $A$ into a diagonal matrix:

**Trick:** Diagonalize: $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Here $D$ is the matrix of eigenvalues and $P$ is the matrix of eigenvectors

**STEP 4:** The rest is just some algebra

$$\begin{aligned} \mathbf{x}'(t) &= A\,\mathbf{x}(t) \\ \mathbf{x}'(t) &= PDP^{-1}\,\mathbf{x}(t) \\ P^{-1}\left(\mathbf{x}'(t)\right) &= \cancel{P^{-1}}\cancel{P}DP^{-1}\,\mathbf{x}(t) \\ \left(P^{-1}\mathbf{x}\right)'(t) &= D\left(P^{-1}\mathbf{x}\right)(t) && P^{-1} \text{ is like a constant} \\ \mathbf{y}'(t) &= D\mathbf{y}(t) && \text{where } \mathbf{y} = P^{-1}\mathbf{x} \end{aligned}$$

So in fact we transformed $\mathbf{x}'(t) = A\mathbf{x}(t)$ into the **DIAGONAL** system $\mathbf{y}'(t) = D\mathbf{y}(t)$, which is precisely the system in **STEP 2:**

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{t} \\ C_2 e^{2t} \end{bmatrix}$$

**STEP 5:** Solve for $\mathbf{x}(t)$

$$\mathbf{y}(t) = P^{-1}\mathbf{x}(t) \Rightarrow \mathbf{x}(t) = P\mathbf{y}(t) \qquad \text{(Think Peyam ☺)}$$

$$\mathbf{x}(t) = \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} C_1 e^{t} \\ C_2 e^{2t} \end{bmatrix}}_{\mathbf{y}(t)} = C_1 e^{1t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \textbf{TA-DAA!!!}$$

**Moral:** Witness here the power of linear algebra. Diagonalization effectively decouples the system $\mathbf{x}'(t) = A\mathbf{x}(t)$ by turning it into a diagonal system $\mathbf{y}'(t) = D\mathbf{y}(t)$ which is much easier to solve.

## 3. Phase Portraits

**Example 2:**

Solve $\mathbf{x}' = A\mathbf{x}$ and draw the phase portrait, where

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

**STEP 1: Eigenvalues**

$$\begin{aligned} |A - \lambda I| = & \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} \\ = & (1-\lambda)(1-\lambda) - (3)(3) \\ = & (1-\lambda)^2 - 9 = 0 \end{aligned}$$

$$(1-\lambda)^2 = 9 \Rightarrow 1 - \lambda = 3 \text{ or } 1 - \lambda = -3$$

Which gives $\lambda = -2$ or $\lambda = 4$

**STEP 2:** $\lambda = -2$

$$\begin{aligned}\mathrm{Nul}(A-(-2)I) &= \left[\begin{array}{cc|c} 1-(-2) & 3 & 0 \\ 3 & 1-(-2) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ &\to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]\end{aligned}$$

$x+y=0 \Rightarrow y=-x$ and so

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = -2 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

**STEP 3:** $\lambda = 4$

$$\begin{aligned}\mathrm{Nul}(A-4I) &= \left[\begin{array}{cc|c} 1-4 & 3 & 0 \\ 3 & 1-4 & 0 \end{array}\right] \\ &= \begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \\ &\xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ &\xrightarrow{(\div -3)R_1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]\end{aligned}$$

$x-y=0 \Rightarrow x=y$ and so

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 4 \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

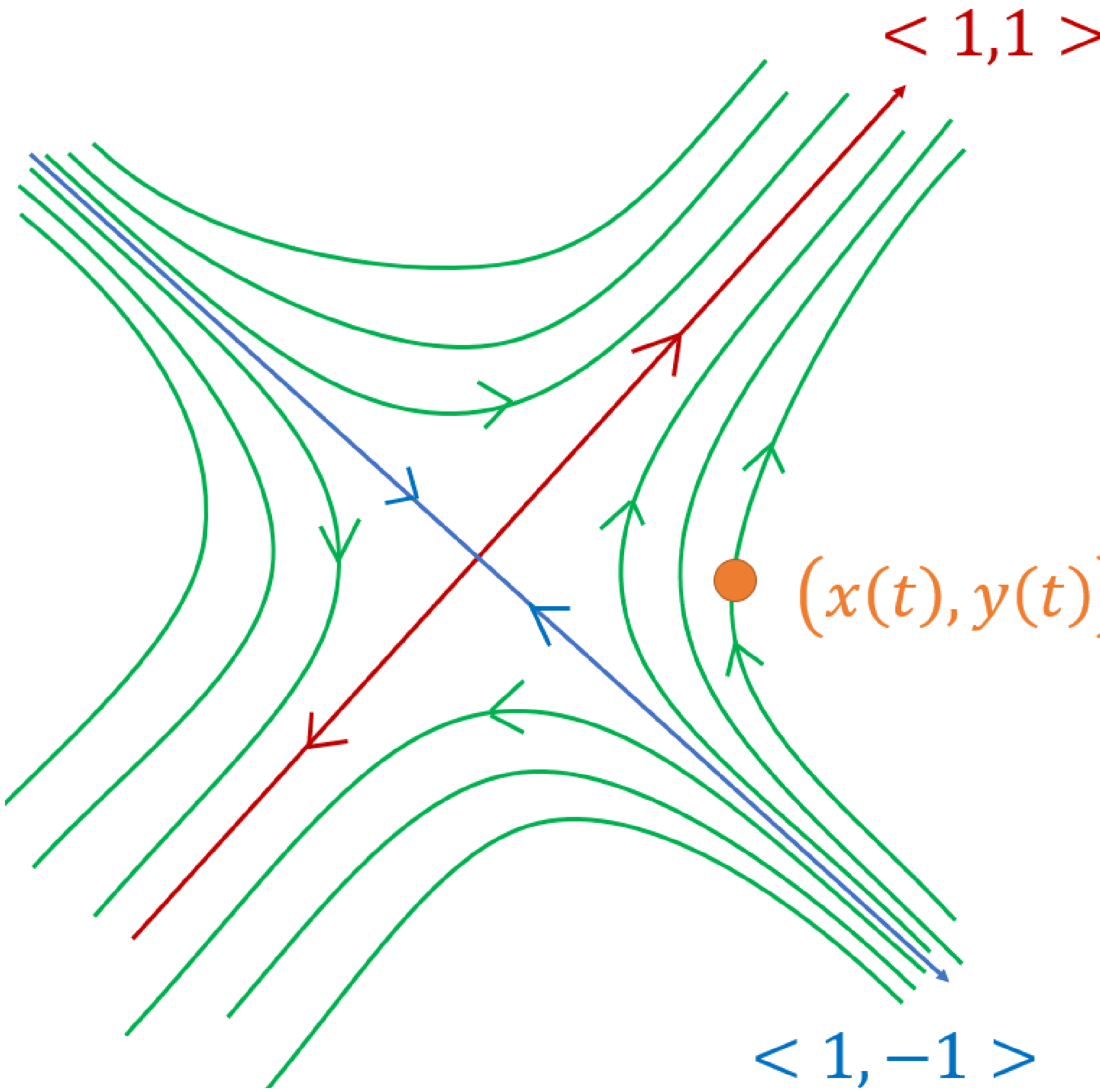
**STEP 4: Solution:**

The eigenvalues are $-2$ and $4$ with corresponding eigenvectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so the solution is

$$\mathbf{x}(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**STEP 5: Phase Portrait**

The cool thing is that we can actually draw out a plot of the solutions!




**Method:**

- First draw the axes with directions $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (eigenvectors)

- On the axis $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ draw arrows going to the origin. This is because $e^{-2t} \to 0$ so solutions on that axis move towards the origin.

- On the axis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ draw arrows going away from the origin. This is because $e^{4t} \to \infty$, so solutions on that axis move away from the origin.

- Finally, for the solutions in between, you just follow the arrows.

**Example 3:**

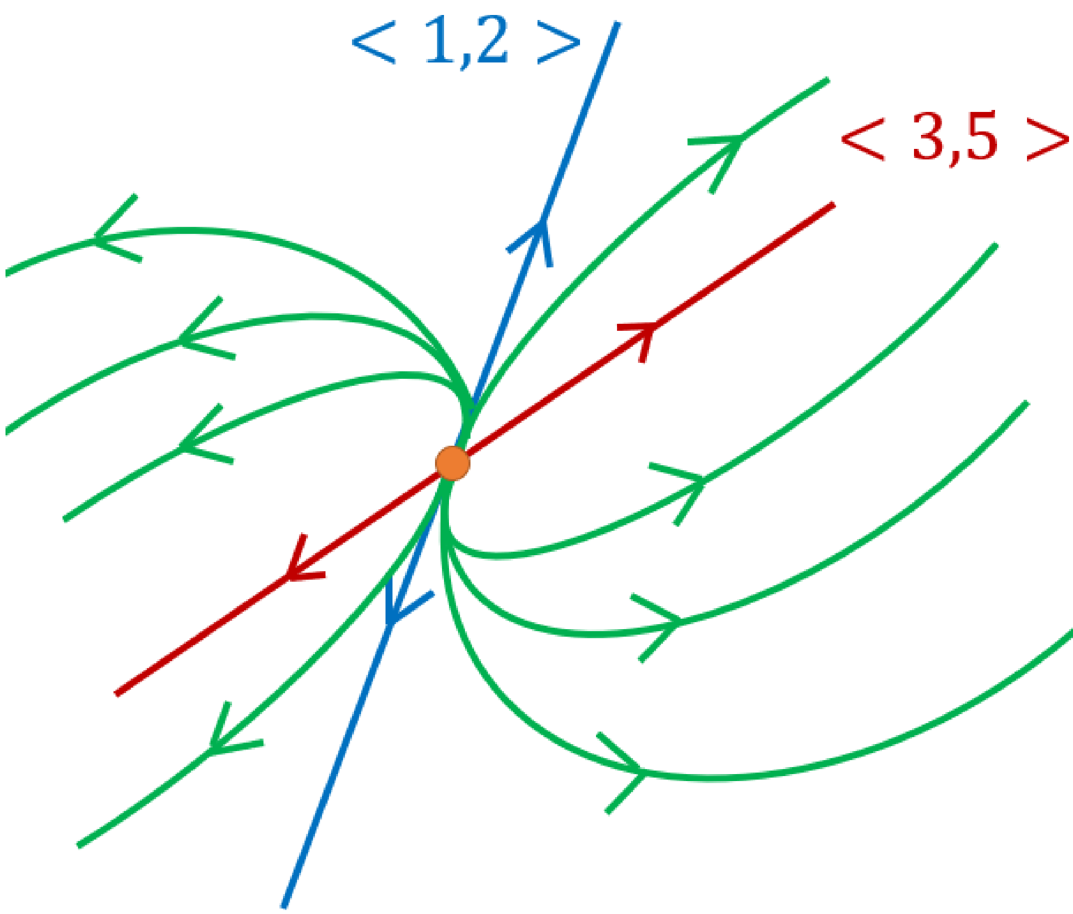
Draw the phase portrait of $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$$

This is the system from before, and we found

$$\mathbf{x}(t) = C_1 e^{1t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Here on each axis, you draw arrows away from the origin (since the eigenvalues are both positive).




**Note:** Since $e^{2t}$ is much bigger than $e^t$, for large $t$, the solutions will look like $C_2 e^{2t} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, which are parallel to $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ This explains the bending shape above

## 4. Initial Conditions

Just like usual, we can have initial conditions

**Example 4:**

$$\mathbf{x}' = A\mathbf{x} \text{ where } A = \begin{bmatrix} 10 & -4 \\ 12 & -4 \end{bmatrix} \text{ and } \mathbf{x}(0) = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

**STEP 1: Eigenvalues**

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 10-\lambda & -4 \\ 12 & -4-\lambda \end{vmatrix} \\ &= (10-\lambda)(-4-\lambda) - 12(-4) \\ &= -40 - 10\lambda + 4\lambda + \lambda^2 + 48 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 2)(\lambda - 4) = 0 \end{aligned}$$

Which gives $\lambda = 2$ or $\lambda = 4$

**STEP 2:** $\lambda = 2$

$$\begin{aligned} \text{Nul}(A - 2I) &= \left[\begin{array}{cc|c} 10-2 & -4 & 0 \\ 12 & -4-2 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 8 & -4 & 0 \\ 12 & -6 & 0 \end{array}\right] \\ &\xrightarrow{(\div -4)R_1\ (\div -6)R_2} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \\ &\to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$2x - y = 0 \Rightarrow y = 2x$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda = 2 \rightsquigarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

**STEP 3:** $\lambda = 4$

$$\begin{aligned} \text{Nul}(A - 4I) &= \left[\begin{array}{cc|c} 10-4 & -4 & 0 \\ 12 & -4-4 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 6 & -4 & 0 \\ 12 & -8 & 0 \end{array}\right] \\ &\xrightarrow{(\div 2)R_1 \ (\div 4)R_2} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right] \\ &\to \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$3x - 2y = 0$. For example $x = 2$ and $y = 3$ satisfy this, and so

$$\lambda = 4 \rightsquigarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

**STEP 4: Solution:**

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

**STEP 5: Initial Condition**

$$\mathbf{x}(0) = C_1 e^0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^0 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Hence we need to solve the system

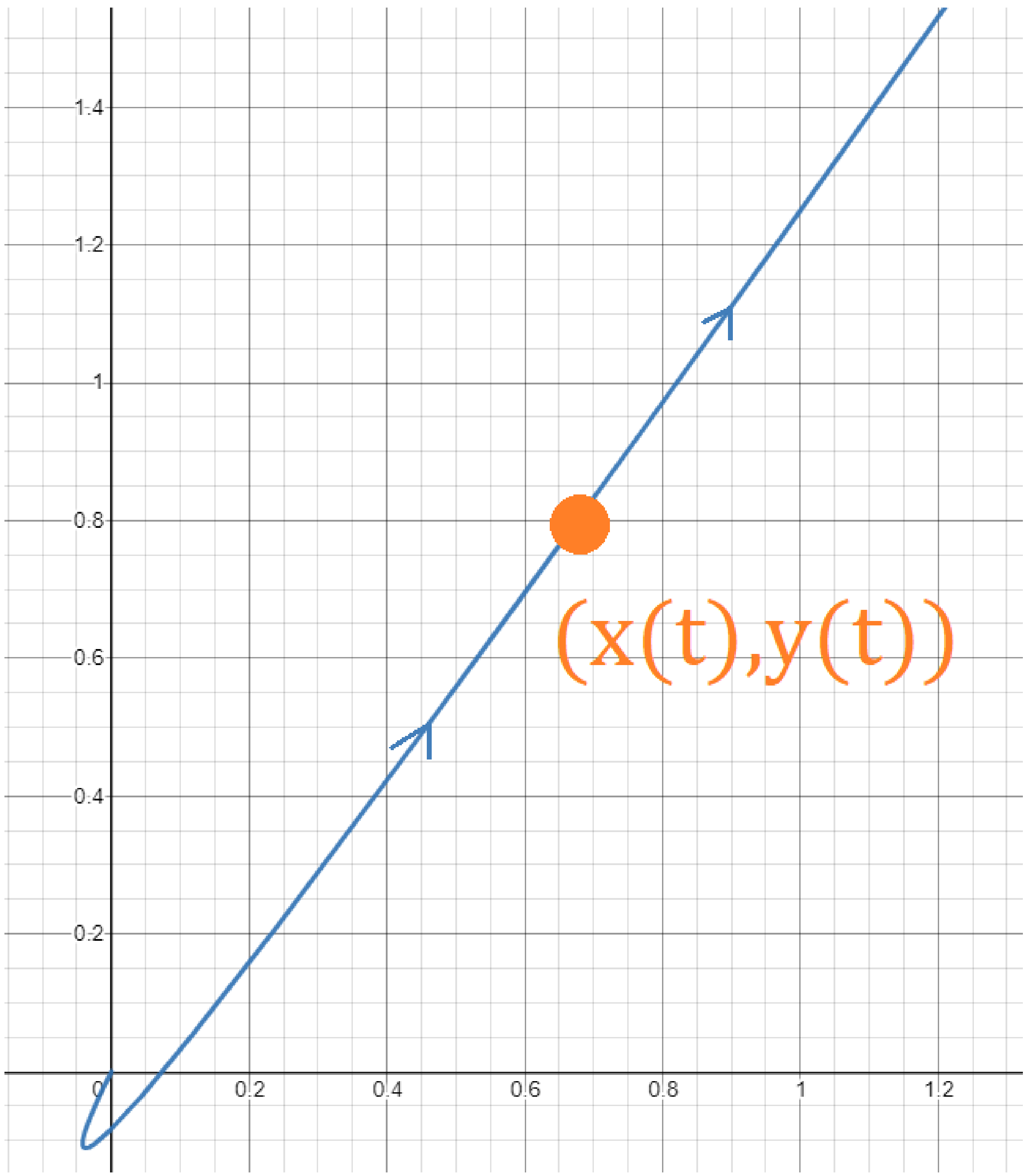
$$\begin{cases} C_1 + 2C_2 = 5 \\ 2C_1 + 3C_2 = 7 \end{cases}$$

$$\left[\begin{array}{cc|c}1 & 2 & 5\\ 2 & 3 & 7\end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}1 & 2 & 5\\ 0 & 3-4 & 7-10\end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|c}1 & 2 & 5\\ 0 & -1 & -3\end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|c}1 & 2 & 5\\ 0 & 1 & 3\end{array}\right]$$
$$\xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c}1 & 0 & 5-2(3)\\ 0 & 1 & 3\end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & 3\end{array}\right]$$

Hence $C_1 = -1$ and $C_2 = 3$ and so

$$\mathbf{x}(t) = -e^{2t}\begin{bmatrix}1\\2\end{bmatrix} + 3e^{4t}\begin{bmatrix}2\\3\end{bmatrix}$$

The solution starts out at $\begin{bmatrix}0\\0\end{bmatrix}$ (at $t = -\infty$) goes down and then up, passes through the initial condition $\begin{bmatrix}5\\7\end{bmatrix}$ at $t = 0$ and eventually becomes parallel to $\begin{bmatrix}2\\3\end{bmatrix}$ see (optional) picture below.


(x(t),y(t))
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1
1.2