LECTURE: QUALITATIVE METHODS

1. Introduction

Today: Focus ODE of the form $y' = f(y)$, called **autonomous ODE**, where the right-hand-side doesn't depend on t

Main Observation:

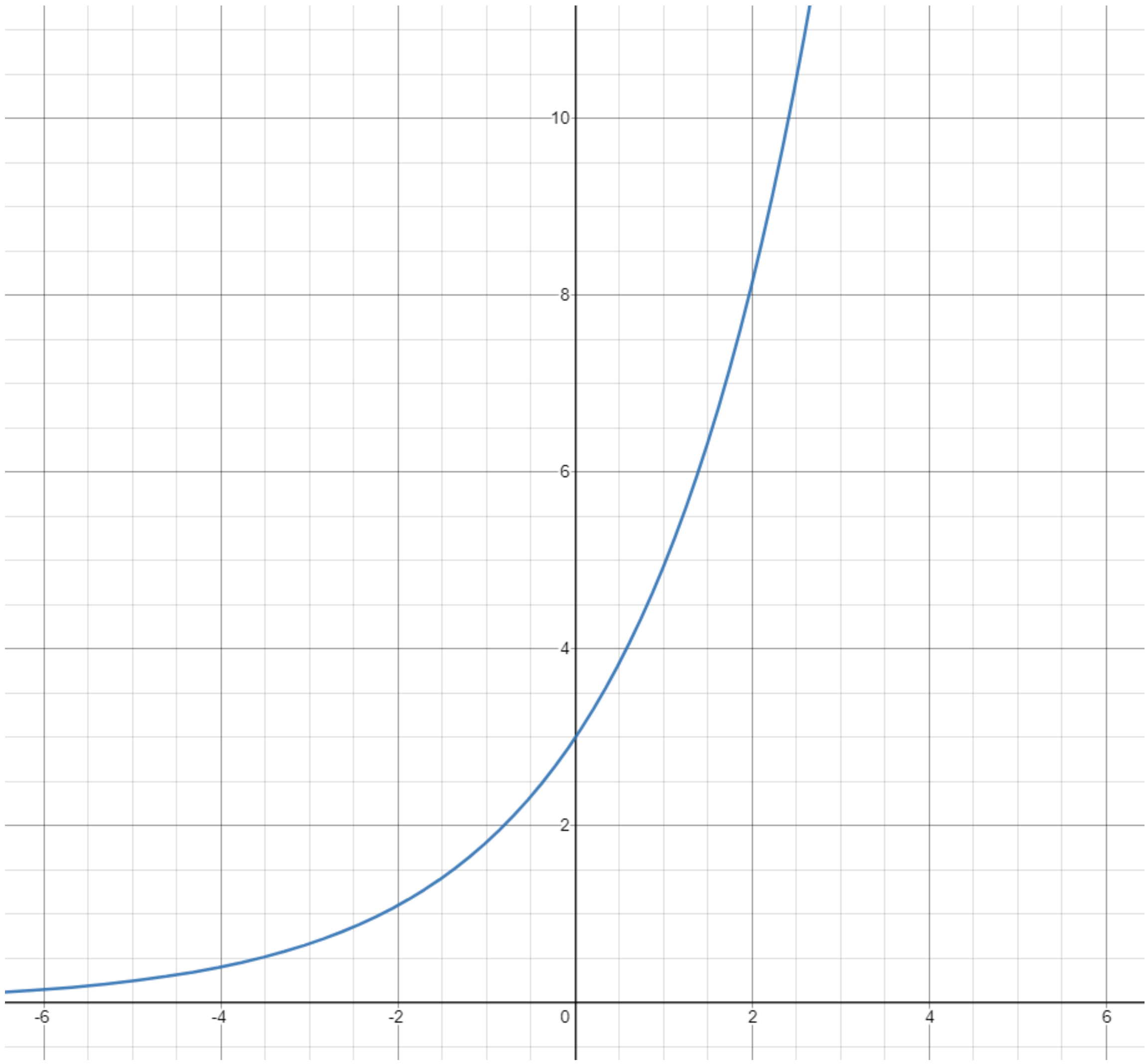
- If $f(y) > 0$ then $y' = f(y) > 0$ and so y is increasing.
- If $f(y) < 0$, then $y' < 0$ so y is decreasing
- Finally, if $f(c) = 0$ for some constant c, then $y = c$ is a solution

$$\text{Because } y' = (c)' = 0 = f(c) = f(y)\checkmark$$

In other words, to study those ODE, we just need to figure out the sign of the right hand side

2. The Logistic Equation

Motivation: A basic model for bacteria growth is $y' = 3y \Rightarrow y = Ce^{3t}$



But to be honest, this model is pretty bad: We can't expect bacteria to grow indefinitely because the amount of food needed for survival is limited.

Here is a more accurate model:

Example 1: (Logistic Equation)

$$y' = 3y\left(1 - \frac{y}{20}\right)$$

Here 3 is the growth rate as before, and 20 is the **limiting factor** (carrying capacity), which will be explained below.

Intuitively: This equation makes sense, because if $y \approx 0$ then we get $y' = 3y$, which is the basic model, but if $y \approx 20$, then we get $y' \approx 0$ because population grows more slowly.

This is of the form $y' = f(y)$ where $f(y) = 3y\left(1 - \frac{y}{20}\right)$

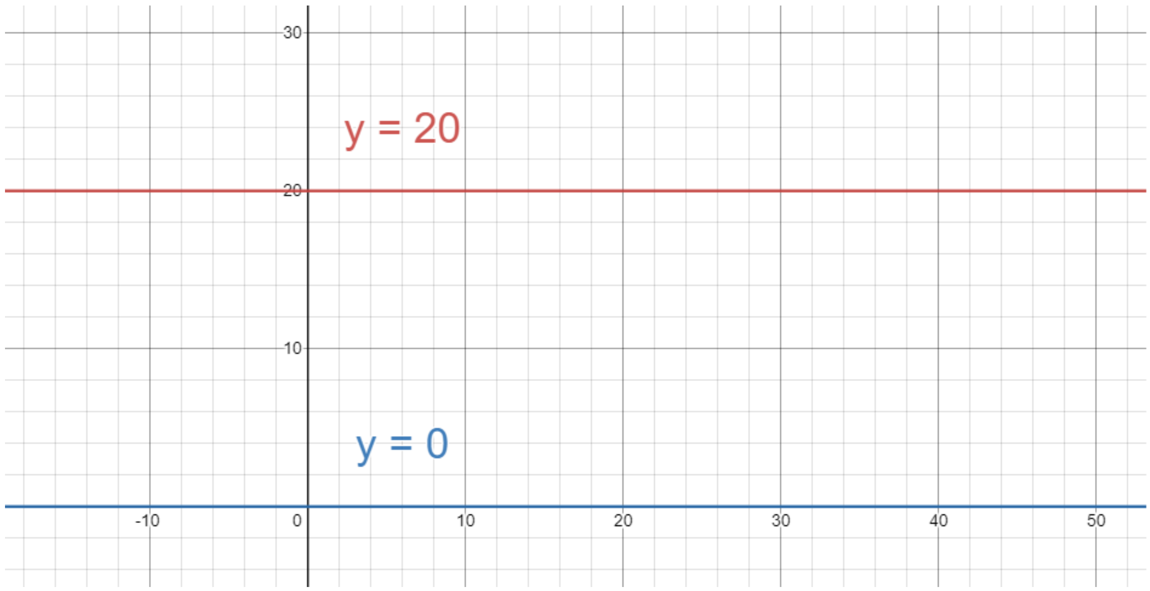
(a) Find the **equilibrium solutions**

This just means set $f(y) = 0$

$$\begin{aligned} f(y) =& 0 \\ 3y\left(1 - \frac{y}{20}\right) =& 0 \\ & y = 0 \text{ or } 1 - \frac{y}{20} = 0 \\ & y = 0 \text{ or } y = 20 \end{aligned}$$

Therefore $y = 0$ and $y = 20$ are solutions.

Interpretation: If the population starts at 20, then it stays at 20



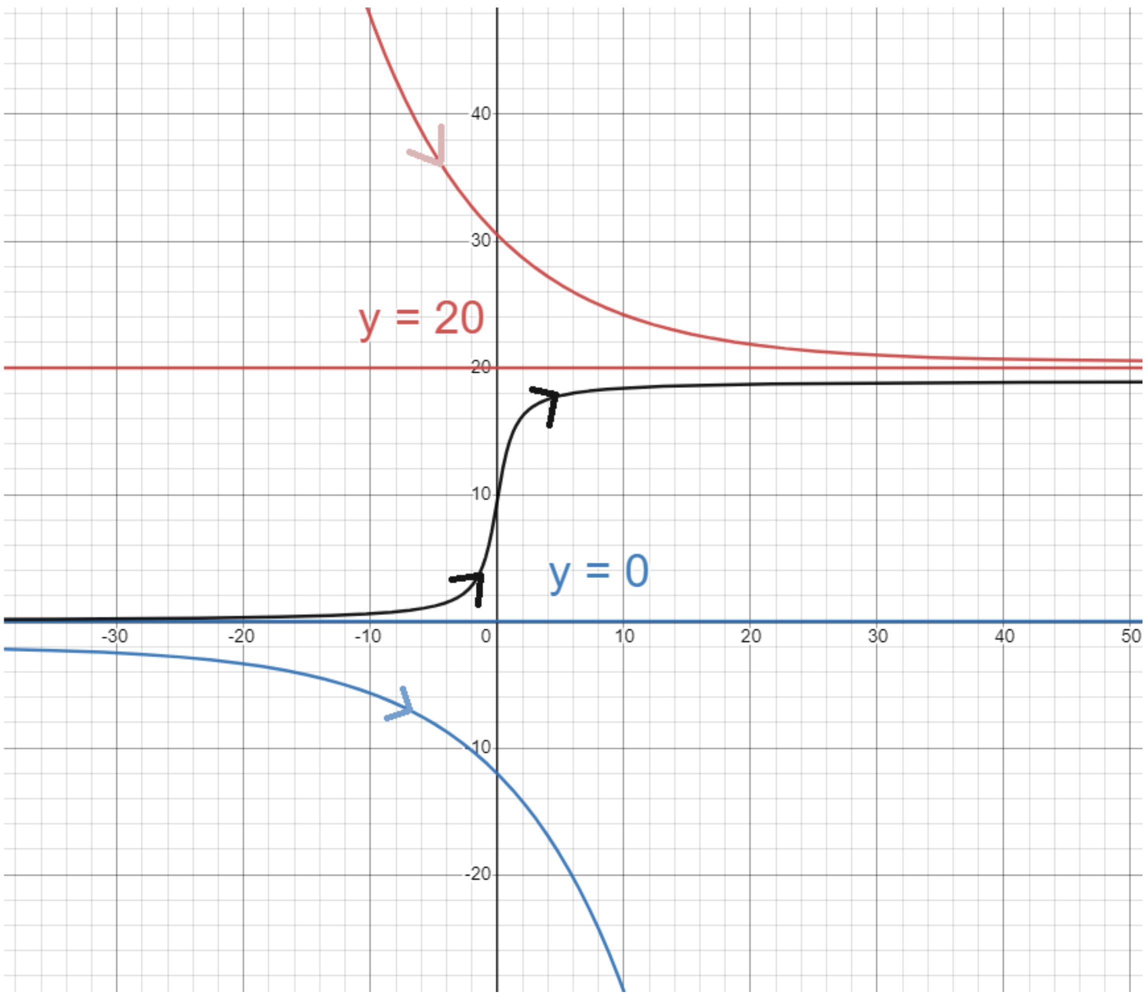
What happens between those values? This is where a bifurcation diagram can help us

(b) Draw a **bifurcation diagram** for this differential equation

Exactly the same as the sign table you used in Calc to draw graphs

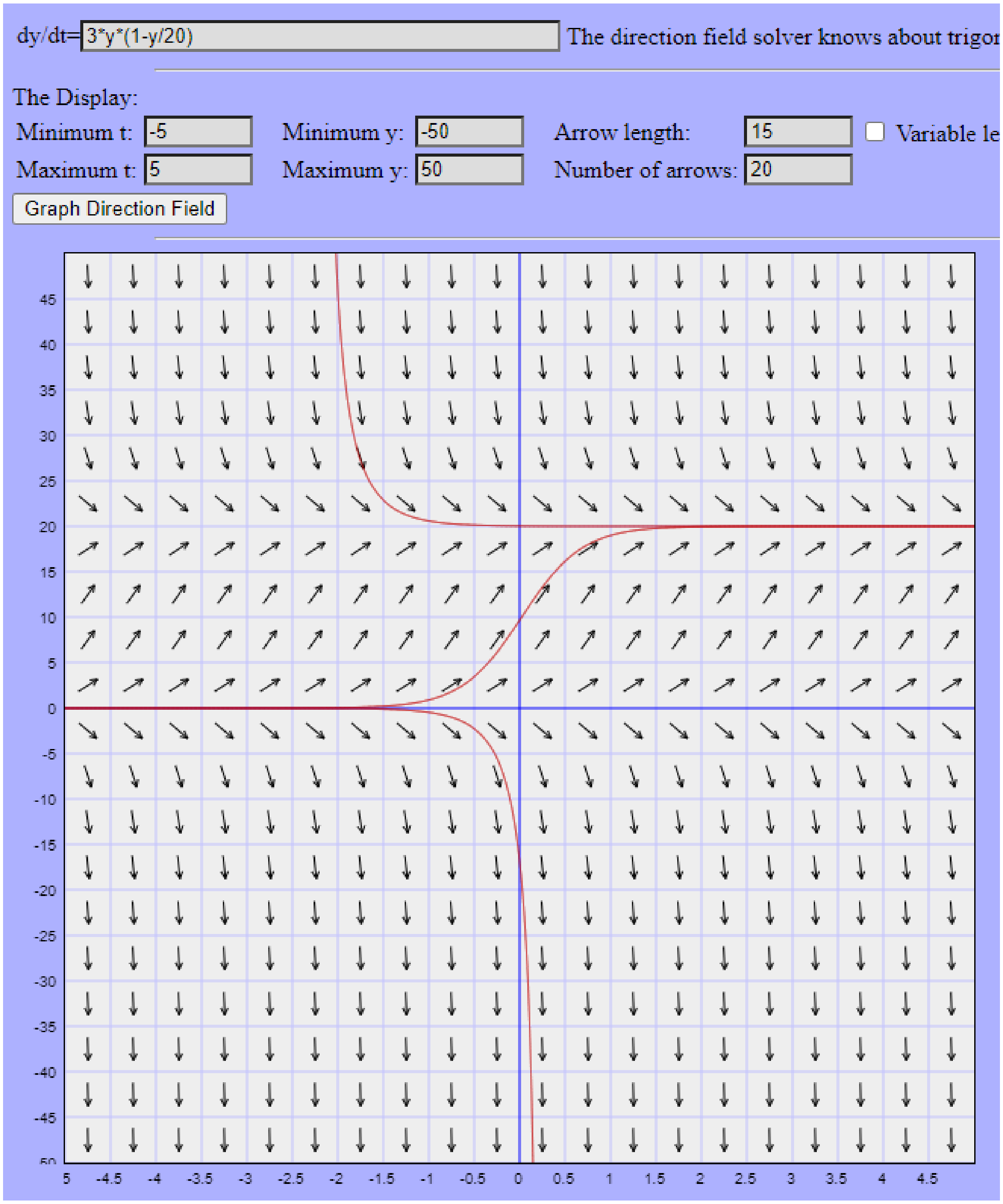
y	$-\infty$		0		20		∞
$3y$		$-$	0	$+$		$+$	
$1 - y/20$		$+$		$+$	0	$-$	
$y' = 3y(1 - y/20)$		$-$	0	$+$	0	$-$	
y		↘		↗		↘	

Interpretation: If y is below 0, then it decreases/moves away from the $y = 0$ solution, but y is between 0 and 20, then it increases/moves towards the $y = 20$ solution. Finally if y is bigger than 20, then it decreases/moves towards the $y = 20$ solution:



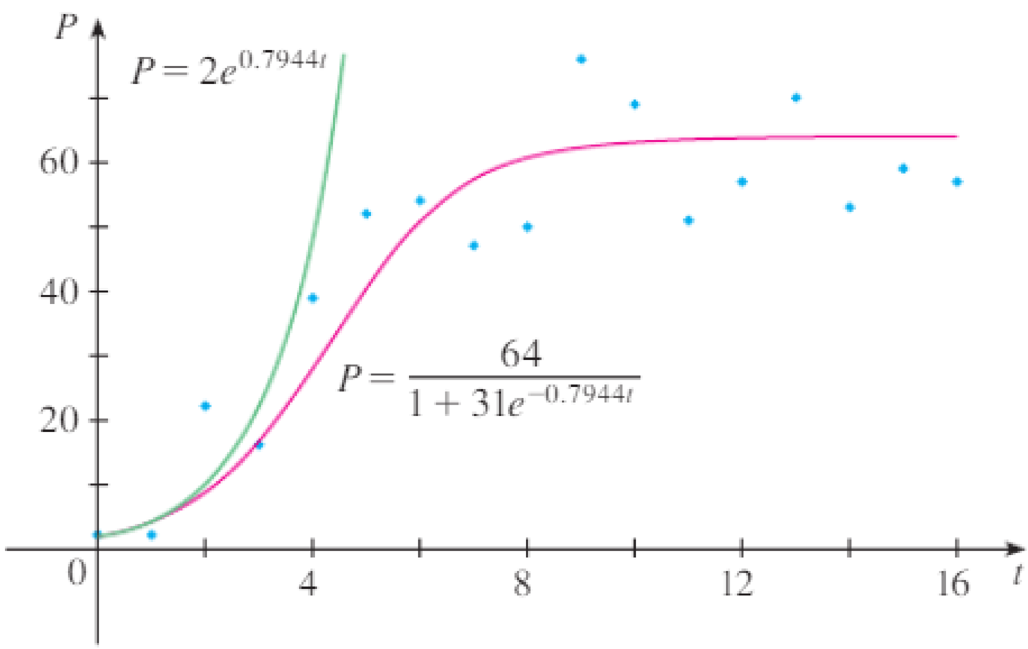
We say that $y = 20$ is a **stable** equilibrium but $y = 0$ is **unstable** (since solutions move away from it)

In fact, a direction field confirms our guesses:

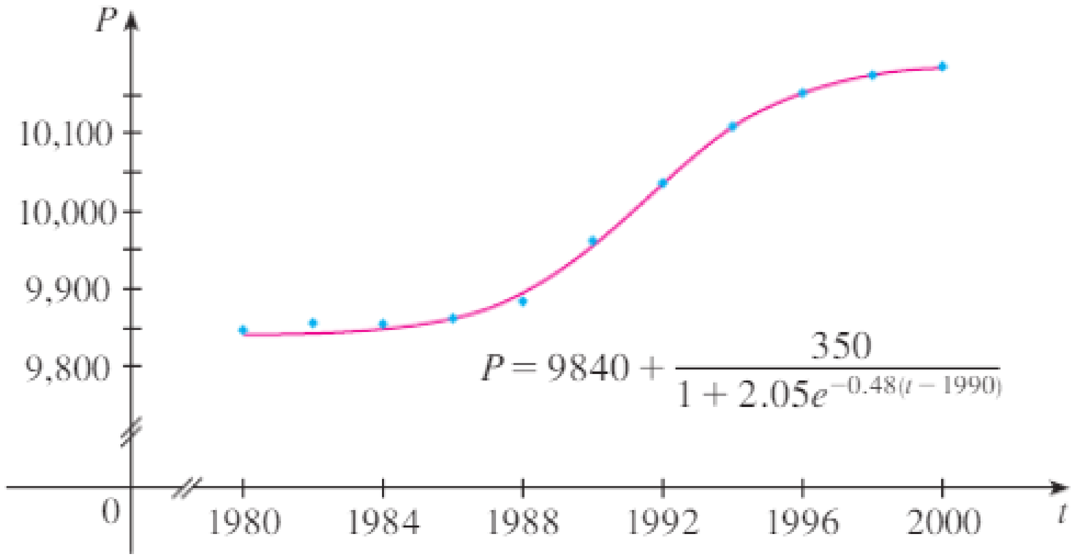

dy/dt= 3*y*(1-y/20)
The direction field solver knows about trigo
The Display:
Minimum t: -5
Maximum t: 5
Minimum y: -50
Maximum y: 50
Arrow length: 15
Number of arrows: 20
Variable le
Graph Direction Field
45
40
35
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
5
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5

Bacteria Interpretation: If the bacteria population is between 0 and 20, the bacteria will multiply until they reach 20 bacteria, which is the limiting factor. They don't reproduce more because of limited resources. Similarly, if the bacteria population is above 20, then the population actually decreases towards 20 because of limited resources. This makes more sense than just saying "it increases exponentially."

Real-life Evidence: The following graph, taken from section 9.4 of Stewart's calculus book, describes the population of bacteria. Notice that the exponential model (in green) is not a good fit, but the logistic model (in red) is a much better fit



Even better, the following graph, taken from the same source, describes the population of Belgium. The logistic model fits like a charm. What is happening here is that the Belgian population was increasing exponentially, but the population eventually slowed down.



3. More Practice

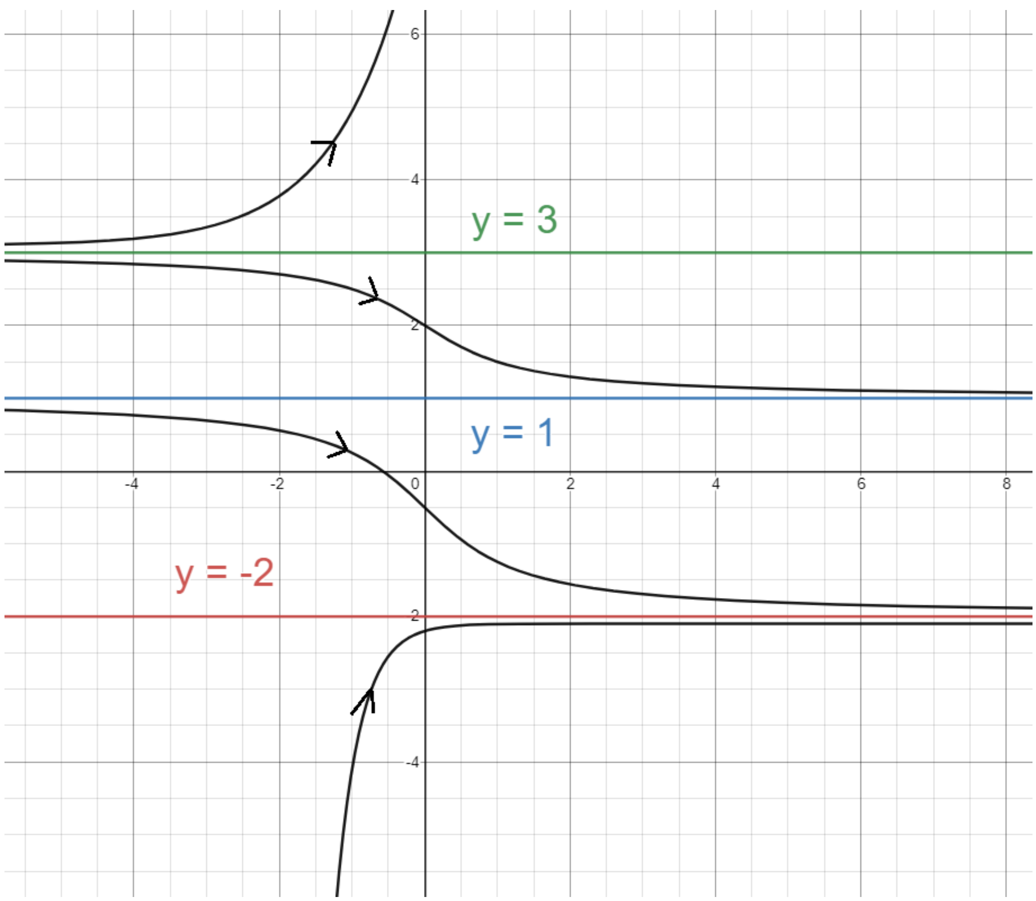
Example 2:

Same questions but for $y' = (y+2)(y-1)^2(y-3)$

$$y' = 0 \Rightarrow (y+2)(y-1)^2(y-3) = 0 \Rightarrow y = -2 \text{ or } y = 1 \text{ or } y = 3$$

Bifurcation Diagram

y	$-\infty$		-2		1		3		∞
$y+2$		$-$	0	$+$		$+$		$+$	
$(y-1)^2$		$+$		$+$	0	$+$		$+$	
$y-3$		$-$		$-$		$-$	0	$+$	
y'		$+$	0	$-$	0	$-$	0	$+$	
y		↗		↘		↘		↗	

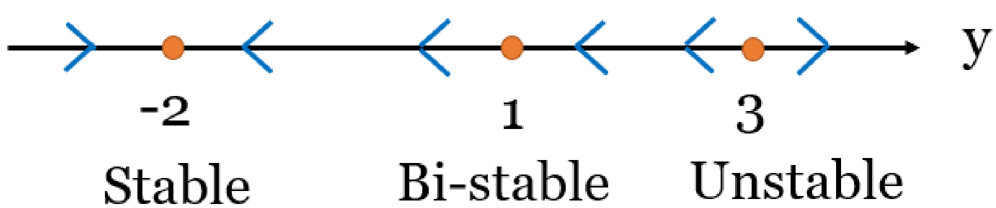


From the picture, it looks like $y = -2$ is stable and $y = 3$ is unstable.

Note: $y = 1$ has a peculiar situation where solutions move towards it from one side and out of it from the other side. This is called **bistable** (or semistable).

Alternative Representation

You can also sometimes represent the solution as follows:



It makes sense if you think of y as being a particle on a line: It gets attracted by -2 (stable) but repulsed by 3 (unstable) and attracted/repulsed by 1 (bistable)

4. Existence/Uniqueness

Recall: Basic ODE

$$\begin{cases} y' = 3y \\ y(0) = 5 \end{cases}$$

Whose solution is $y(t) = 5e^{3t}$

Notice two important things here:

(1) The ODE *has* a solution, namely $y = 5e^{3t}$ (Existence)

(2) There are no other solutions. If y is another solution of the ODE, then we necessarily have $y = 5e^{3t}$ (Uniqueness)

To see what can go wrong, let's look at the following example:

5. A Crazy Example

Video: Differential Equation with a Twist

Example 3:

$$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$$

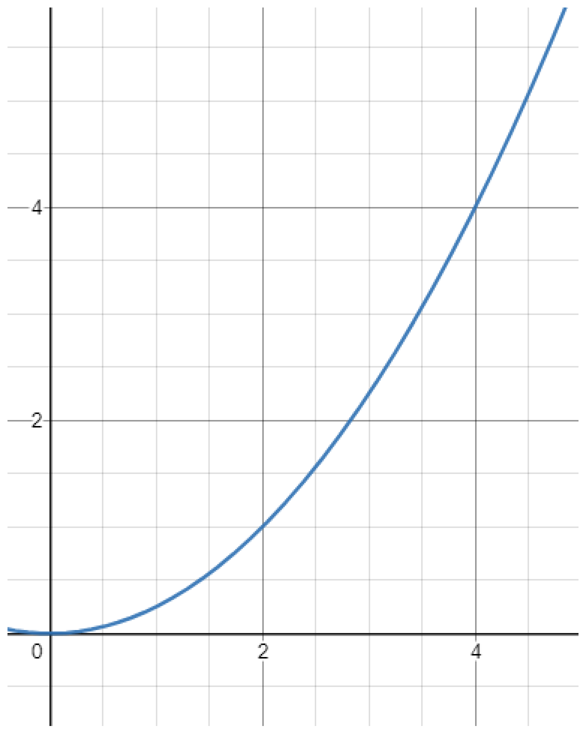
here assume $t \geq 0$

$$\begin{aligned} y' &= \sqrt{y} \\ \frac{y'}{\sqrt{y}} &= 1 \\ (2\sqrt{y})' &= 1 \\ 2\sqrt{y} &= t + C \\ \sqrt{y} &= \frac{t+C}{2} \\ y &= \left(\frac{t+C}{2}\right)^2 \end{aligned}$$

$$y(0) = \left(\frac{C}{2}\right)^2 = \frac{C^2}{4} = 0 \Rightarrow C = 0$$

$$y = \left(\frac{t}{2}\right)^2 = \frac{t^2}{4}$$

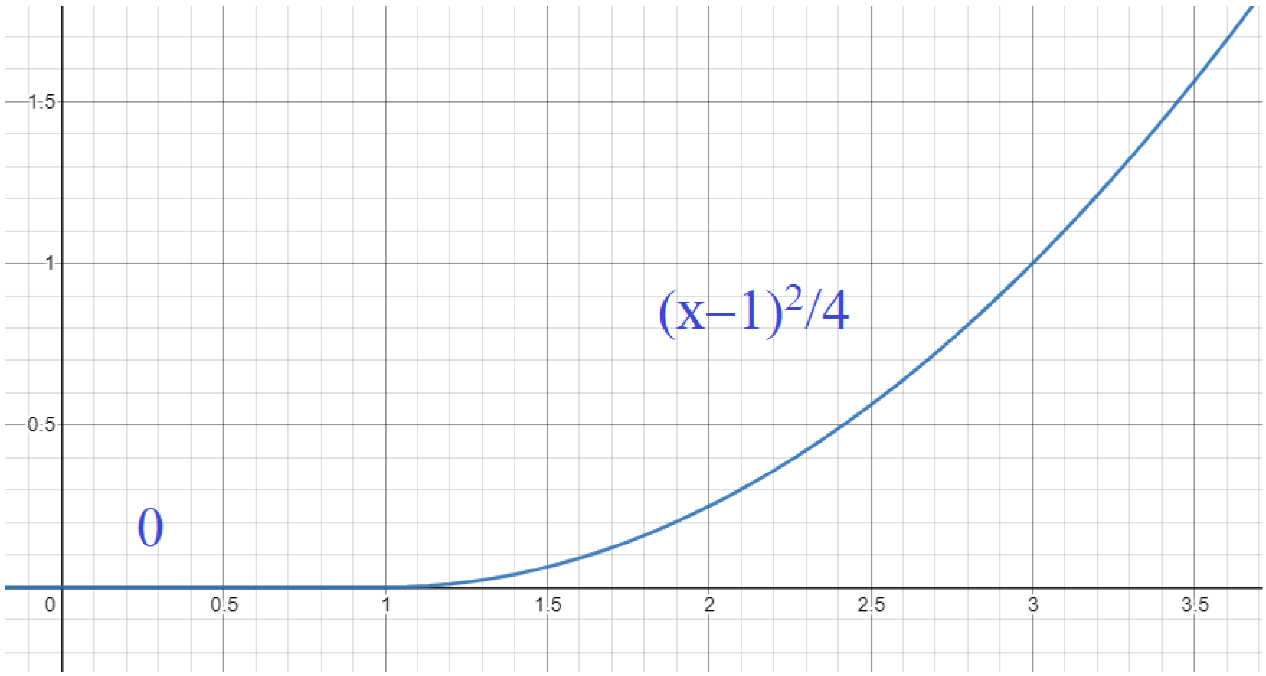
$$\textbf{Check:} \quad y' = \frac{2t}{4} = \frac{t}{2} \text{ and } \sqrt{y} = \sqrt{\frac{t^2}{4}} = \frac{t}{2} \text{ Since } t \geq 0 \checkmark$$



BUT there are other solutions!!! Notice $y = 0$ is also a solution and satisfies $y(0) = 0$

BUUUUUT there are even more solutions!!!!!!!! For example, consider the following function:

$$y = \begin{cases} 0 & \text{if } t \leq 1 \\ \frac{(t-1)^2}{4} & \text{if } t \geq 1 \end{cases}$$



This is also a solution. And in fact, you can replace "1" by any constant $C \geq 0$ and *still* get a solution, so in fact there are **infinitely** many solutions, despite the initial condition $y(0) = 0$

So even though $y' = \sqrt{y}$ is a nice differential equation, there is no unique solution. We will see next time what makes this fail.