LECTURE: COMPLEX EIGENVALUES

1. Complex Eigenvalues

Video: Complex Eigenvalues

Example 1:

Solve $\mathbf{x}' = A\mathbf{x}$ and draw the phase portrait where

$$A = \begin{bmatrix} -2 & 4 \\ -2 & 2 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} \left|A - \lambda I\right| = & \begin{vmatrix} -2-\lambda & 4 \\ -2 & 2-\lambda \end{vmatrix} \\ = & (-2-\lambda)(2-\lambda) - (4)(-2) \\ = & -4 + \cancel{2\lambda} - \cancel{2\lambda} + \lambda^2 + 8 \\ = & \lambda^2 + 4 \\ = & 0 \end{aligned}$$

$$\lambda^2 = -4 \Rightarrow \lambda = \pm\sqrt{-4} = \pm 2i$$

STEP 2: $\lambda = 2i$

$$\text{Nul }(A - (2i)I) = \left[\begin{array}{cc|c} -2-2i & 4 & 0 \\ -2 & 2-2i & 0 \end{array}\right] \xrightarrow{(\div -2R_1)\ (\div -2R_2)} \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{array}\right]$$

Trick: Since $2i$ is an eigenvalue, one row has to be 0[1] Keep one of the rows (doesn't matter which one) and make the other zero:

$$Nul(A - 2iI) = \left[\begin{array}{cc|c} 1+i & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $(1+i)x - 2y = 0$. For example $x = 2$ and $y = 1+i$ works

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

GOOD NEWS: We won't need the other eigenvalue $\lambda = -2i$ Just like for complex roots, one complex eigenvalue will give us all the solutions.

If you're interested though, an eigenvector for $\lambda = -2i$ would be

$$\overline{\mathbf{v}} = \begin{bmatrix} 2 \\ 2-i \end{bmatrix} \text{ (Complex conjugate)}$$

STEP 3: Solution

Observation: Since $\lambda = 2i \rightsquigarrow \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$ this tells us that

$$e^{\lambda t}\mathbf{v} = e^{(2i)t} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} \text{ is a solution}$$

[1] This follows from Linear Algebra: otherwise there would be 2 pivots, so the matrix would be invertible, which contradicts that $2i$ is an eigenvalue

Split this up into real and imaginary parts:

$$
\begin{aligned}
e^{(2i)t}\begin{bmatrix}2\\1+i\end{bmatrix} =& e^{i(2t)}\begin{bmatrix}2+0i\\1+i\end{bmatrix}\\
&= (\cos(2t)+i\sin(2t))\left(\begin{bmatrix}2\\1\end{bmatrix}+i\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \cos(2t)\begin{bmatrix}2\\1\end{bmatrix}+\cos(2t)\begin{bmatrix}0\\1\end{bmatrix}i+\sin(2t)\begin{bmatrix}2\\1\end{bmatrix}i+\sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\underbrace{i^2}_{-1}\\
&= \left(\cos(2t)\begin{bmatrix}2\\1\end{bmatrix}-\sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right)+i\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix}+\sin(2t)\begin{bmatrix}2\\1\end{bmatrix}\right)
\end{aligned}
$$

Fact:

Both the real and imaginary parts solve the ODE

So $\cos(2t)\begin{bmatrix}2\\1\end{bmatrix}-\sin(2t)\begin{bmatrix}0\\1\end{bmatrix}$ and $\cos(2t)\begin{bmatrix}0\\1\end{bmatrix}+\sin(2t)\begin{bmatrix}2\\1\end{bmatrix}$ are solutions

Analogy: This is the analog in $y''+y=0$ of finding $\cos(t)$ and $\sin(t)$ from which we eventually obtained $y = A\cos(t)+B\sin(t)$

The same thing is true here:

Solution:

$$
\mathbf{x}(t) = C_1\left(\cos(2t)\begin{bmatrix}2\\1\end{bmatrix}-\sin(2t)\begin{bmatrix}0\\1\end{bmatrix}\right)+C_2\left(\cos(2t)\begin{bmatrix}0\\1\end{bmatrix}+\sin(2t)\begin{bmatrix}2\\1\end{bmatrix}\right)
$$

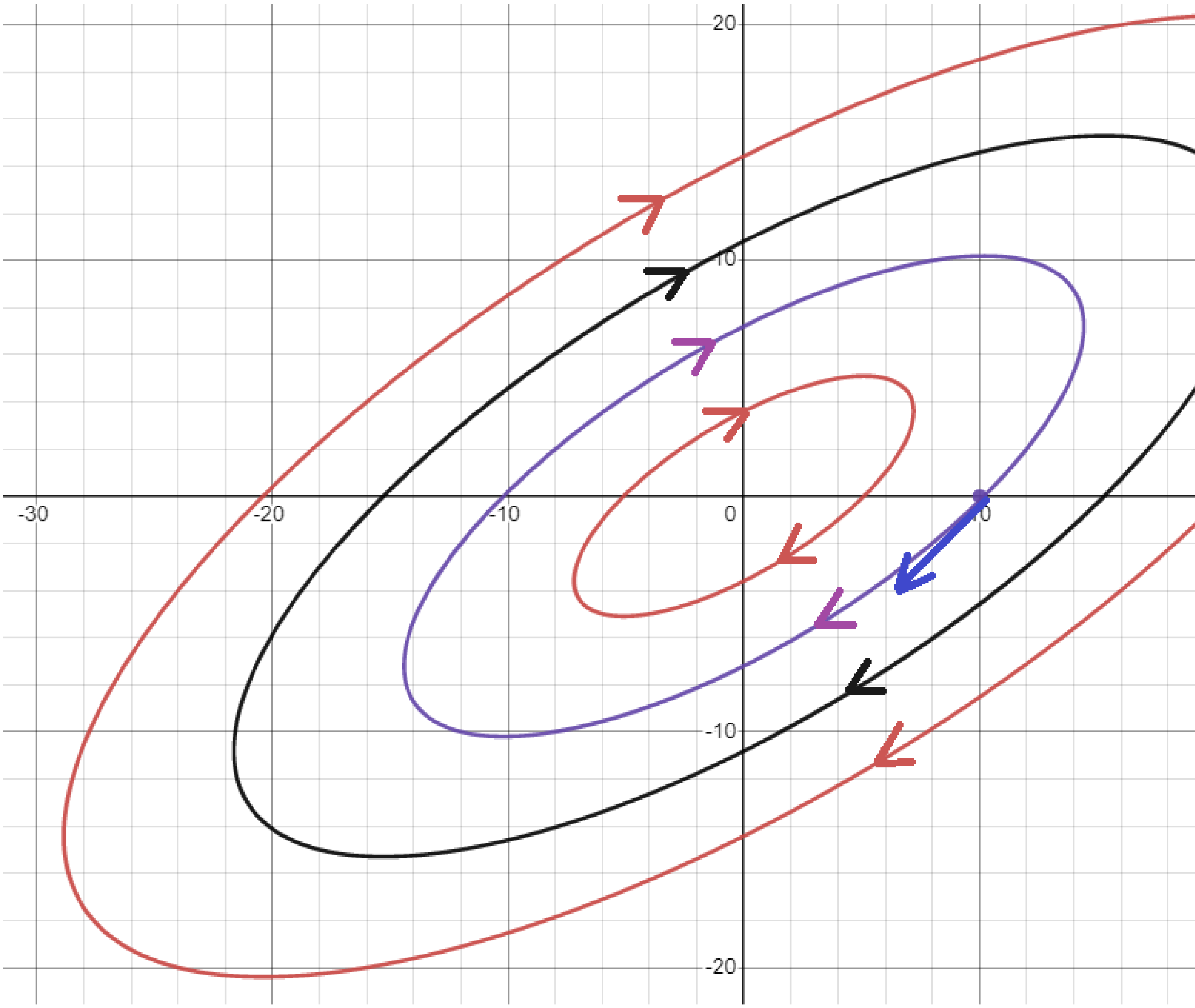
Where C_1 and C_2 are constants

If you want, you can rewrite this as (optional)

$$
\mathbf{x}(t) = C_1\begin{bmatrix}2\cos(2t)\\\cos(2t)-\sin(2t)\end{bmatrix}+C_2\begin{bmatrix}2\sin(2t)\\\cos(2t)+\sin(2t)\end{bmatrix}
$$

STEP 4: Phase Portrait

Because of $\cos(2t)$ and $\sin(2t)$, there is something circular going on, and in fact the solutions here are **ellipses** or circles.



Aside: You're not responsible for finding the axes or the direction for the ellipse. If you're curious though:

To find the direction (clockwise or counterclockwise): Pick any point, say $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then at that point, we have

$$\mathbf{x}'(t) = \begin{bmatrix} -2 & 4 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

So the solution that goes through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ moves in the direction of $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

This tells you that, in the picture above, the ellipses are moving clockwise (the picture illustrates this with the point $(10, 0)$)

To find the axes: (this is a bit trickier): You could in theory solve for $\cos(2t)$ and $\sin(2t)$ in terms of $\mathbf{x}(t)$ by inverting the matrix below, since the solution can be written as

$$\mathbf{x}(t) = \begin{bmatrix} 2C_1 & 2C_2 \\ C_1 + C_2 & -C_1 + C_2 \end{bmatrix} \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$$

And then use $\cos^2(2t) + \sin^2(2t) = 1$ to get an equation of an ellipse. Then use linear algebra, more precisely quadratic forms, to find the axes of the ellipse. For more info about quadratic forms, check out

Video: Quadratic forms

2. More Practice

Example 2:

Solve $\mathbf{x}' = A\mathbf{x}$ and draw the phase portrait where

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} 1-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) - (5)(-2) \\ &= 3 - \lambda - 3\lambda + \lambda^2 + 10 \\ &= \lambda^2 - 4\lambda + 13 \\ &= (\lambda - 2)^2 - 4 + 13 \\ &= (\lambda - 2)^2 + 9 \end{aligned}$$

$$(\lambda - 2)^2 = -9 \Rightarrow \lambda - 2 = \pm 3i \Rightarrow \lambda = 2 \pm 3i$$

STEP 2: $\lambda = 2 + 3i$

$$\begin{aligned} \text{Nul}\,(A - (2+3i)I) &= \left[\begin{array}{cc|c} 1-(2+3i) & 5 & 0 \\ -2 & 3-(2+3i) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{cc|c} -2 & 1-3i & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

Hence $-2x + (1-3i)y = 0$. For example $x = (1-3i)$ and $y = 2$ satisfies this, and so

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$$

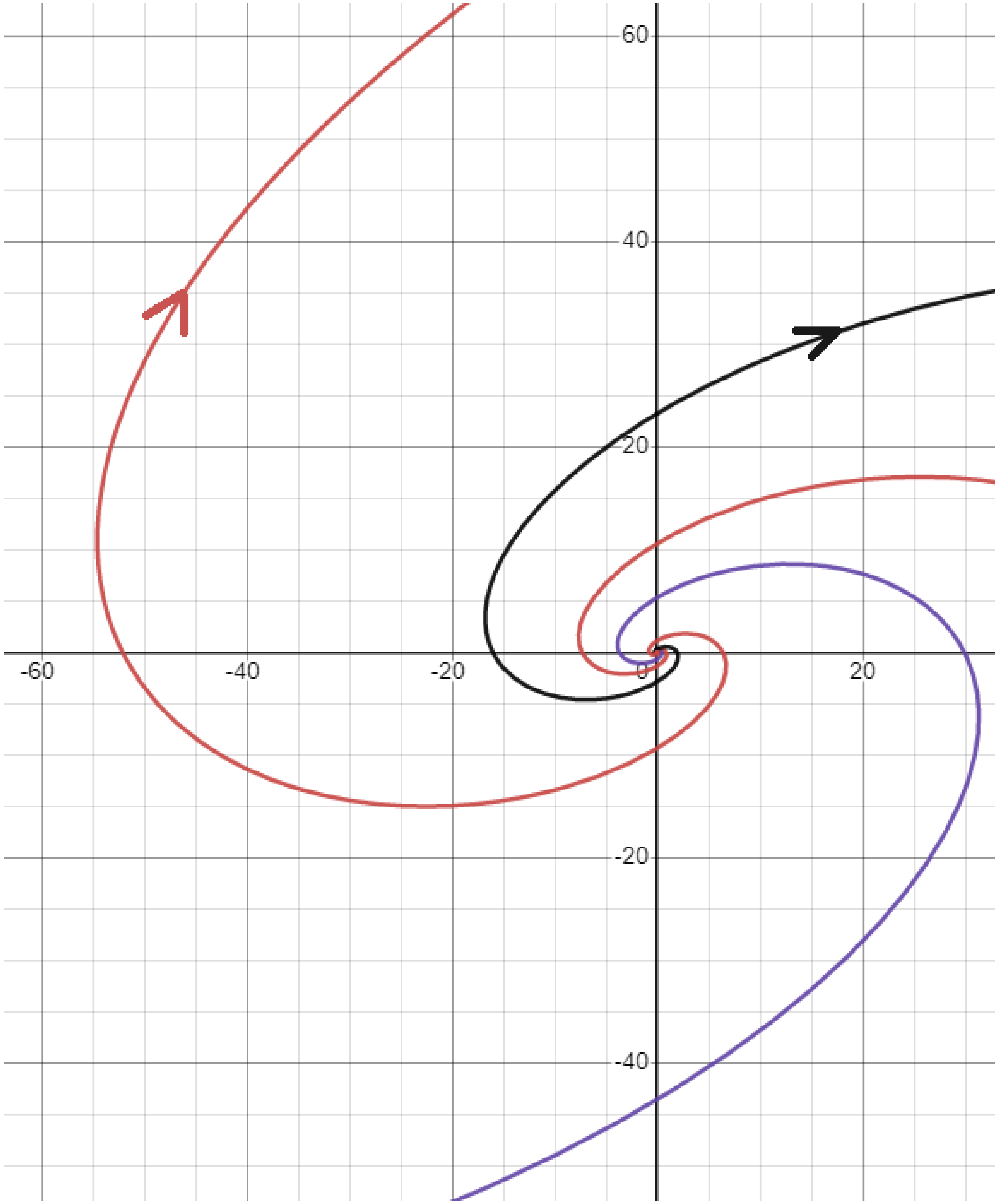
STEP 3: Solution

$$
\begin{aligned}
e^{(2+3i)t}\begin{bmatrix}1-3i\\2\end{bmatrix} &= \left(e^{2t}e^{3ti}\right)\left(\begin{bmatrix}1\\2\end{bmatrix}+i\begin{bmatrix}-3\\0\end{bmatrix}\right)\\
&= e^{2t}\left(\cos(3t)+i\sin(3t)\right)\left(\begin{bmatrix}1\\2\end{bmatrix}+i\begin{bmatrix}-3\\0\end{bmatrix}\right)\\
&= e^{2t}\left(\cos(3t)\begin{bmatrix}1\\2\end{bmatrix}-\sin(3t)\begin{bmatrix}-3\\0\end{bmatrix}\right)\\
&+ ie^{2t}\left(\cos(3t)\begin{bmatrix}-3\\0\end{bmatrix}+\sin(3t)\begin{bmatrix}1\\2\end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}(t) = &C_1e^{2t}\left(\cos(3t)\begin{bmatrix}1\\2\end{bmatrix}-\sin(3t)\begin{bmatrix}-3\\0\end{bmatrix}\right)\\
&+C_2e^{2t}\left(\cos(3t)\begin{bmatrix}-3\\0\end{bmatrix}+\sin(3t)\begin{bmatrix}1\\2\end{bmatrix}\right)
\end{aligned}
$$

STEP 4: Phase Portrait

Because of the e^{2t} term, the solution here is spiraling outwards. If you want, you can once again determine the axes and direction of the spiral the same way you did with the ellipse.


60
40
20
-20
-40
-60
-40
-20
0
20

3. Initial Conditions

Example 3: (more practice)

Solve $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 5 \\ -15 \end{bmatrix}$ where

$$A = \begin{bmatrix} -7 & -5 \\ 5 & -1 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned} \left|A - \lambda I\right| &= \begin{vmatrix} -7-\lambda & -5 \\ 5 & -1-\lambda \end{vmatrix} \\ &= (-7-\lambda)(-1-\lambda) - (-5)(5) \\ &= 7 + 7\lambda + \lambda + \lambda^2 + 25 \\ &= \lambda^2 + 8\lambda + 32 \\ &= (\lambda+4)^2 - 4^2 + 32 \\ &= (\lambda+4)^2 + 16 \end{aligned}$$

$$(\lambda+4)^2 = -16 \Rightarrow \lambda + 4 = \pm 4i \Rightarrow \lambda = -4 \pm 4i$$

STEP 2: $\lambda = -4 + 4i$

$$\begin{aligned} \text{Nul}\,(A - (-4+4i)I) &= \left[\begin{array}{cc|c} -7-(-4+4i) & -5 & 0 \\ 5 & -1-(-4+4i) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -3-4i & -5 & 0 \\ 5 & 3-4i & 0 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{cc|c} 5 & 3-4i & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

Hence $5x + (3-4i)y = 0$. For example, $x = 3 - 4i$ and $y = -5$ works

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3-4i \\ -5 \end{bmatrix}$$

An eigenvector for $\lambda = -4+4i$ is $\begin{bmatrix} 3-4i \\ -5 \end{bmatrix}$

STEP 3: Solution

$$\begin{aligned} e^{(-4+4i)t} \begin{bmatrix} 3-4i \\ -5 \end{bmatrix} =& e^{-4t} \left(\cos(4t) + i\sin(4t)\right) \left(\begin{bmatrix} 3 \\ -5 \end{bmatrix} + i \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) \\ =& e^{-4t} \left(\cos(4t) \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \sin(4t) \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) \\ &+ ie^{-4t} \left(\cos(4t) \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \sin(4t) \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{x}(t) =& C_1 e^{-4t} \left(\cos(4t) \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \sin(4t) \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) \\ &+ C_2 e^{-4t} \left(\cos(4t) \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \sin(4t) \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right) \end{aligned}$$

STEP 4: Initial Condition

$$\begin{aligned} \mathbf{x}(0) =& C_1 e^{0} \left(\cos(0) \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \sin(0) \begin{bmatrix} -4 \\ 0 \end{bmatrix} \right) \\ &+ C_2 e^{0} \left(\cos(0) \begin{bmatrix} -4 \\ 0 \end{bmatrix} + \sin(0) \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right) \\ =& C_1 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + C_2 \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -15 \end{bmatrix} \end{aligned}$$

Here either do Gaussian elimination, or easier to do directly:

$$\begin{cases} 3C_1 - 4C_2 = 5 \\ \quad -5C_1 = -15 \end{cases} \Rightarrow \begin{cases} -4C_2 = 5 - 3(C_1) = 5 - 3(3) = -4 \\ \quad C_1 = 3 \end{cases} \Rightarrow \begin{cases} C_2 = 1 \\ C_1 = 3 \end{cases}$$

Which gives $C_1 = 3$ and $C_2 = 1$

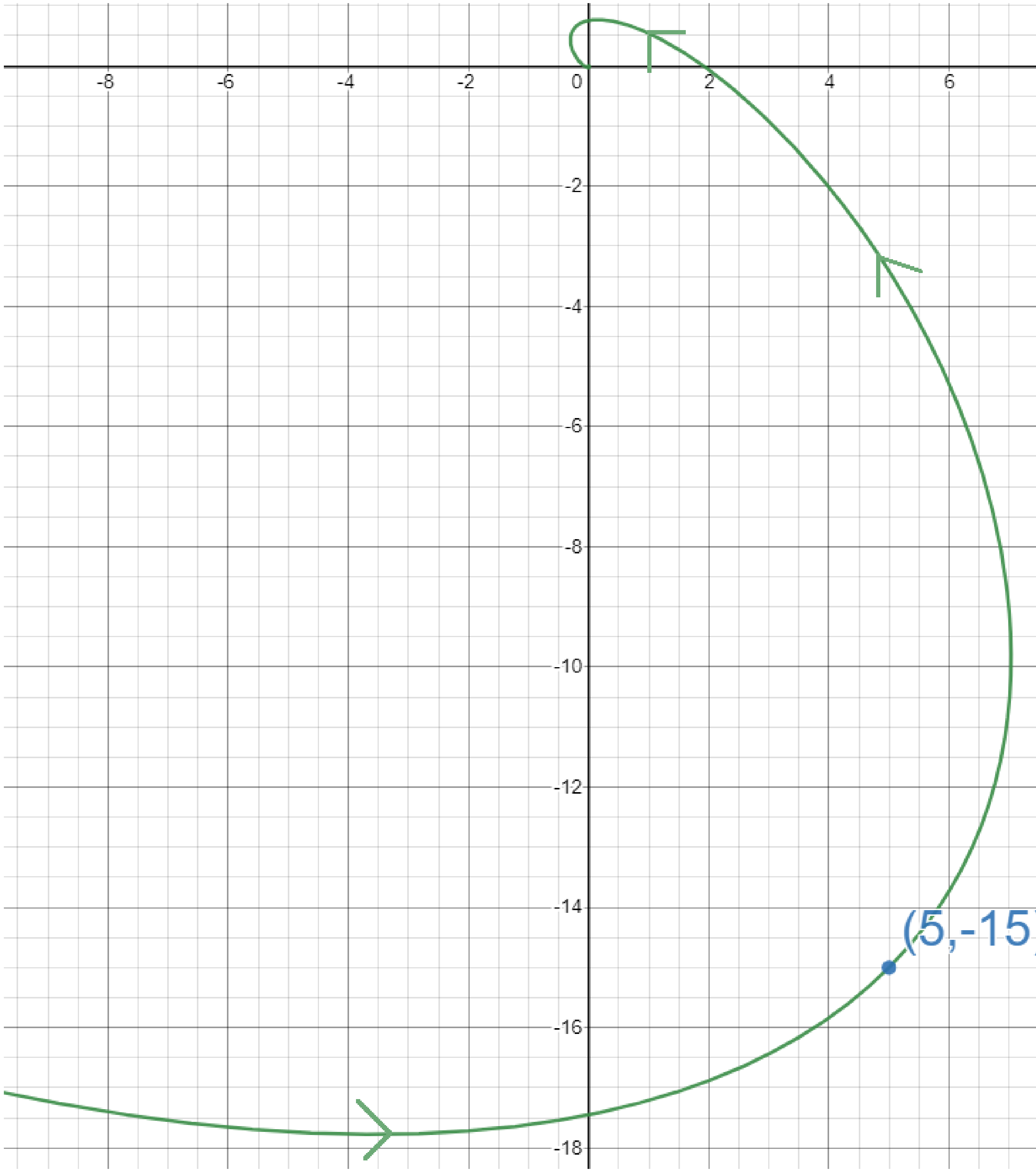
$$\begin{aligned}\mathbf{x}(t) =& 3e^{-4t}\left(\cos(4t)\begin{bmatrix}3\\-5\end{bmatrix} - \sin(4t)\begin{bmatrix}-4\\0\end{bmatrix}\right)\\ &+1e^{-4t}\left(\cos(4t)\begin{bmatrix}-4\\0\end{bmatrix} + \sin(4t)\begin{bmatrix}3\\-5\end{bmatrix}\right)\end{aligned}$$

Which, (optional) if you want, after simplification, you can rewrite as

$$\mathbf{x}(t) = e^{-4t}\begin{bmatrix}5\cos(4t) + 15\sin(4t)\\ -15\cos(4t) - 5\sin(4t)\end{bmatrix}$$

Phase Portrait:

Because of the e^{-4t} term, all the solutions spiral into $\begin{bmatrix}0\\0\end{bmatrix}$ which is the opposite of the previous problem.


-8
-6
-4
-2
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2
4
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-2
-4
-6
-8
-10
-12
-14
-16
-18
(5,-15)