LECTURE: SEPARABLE EQUATIONS

1. Separation of Variables

Example 1:

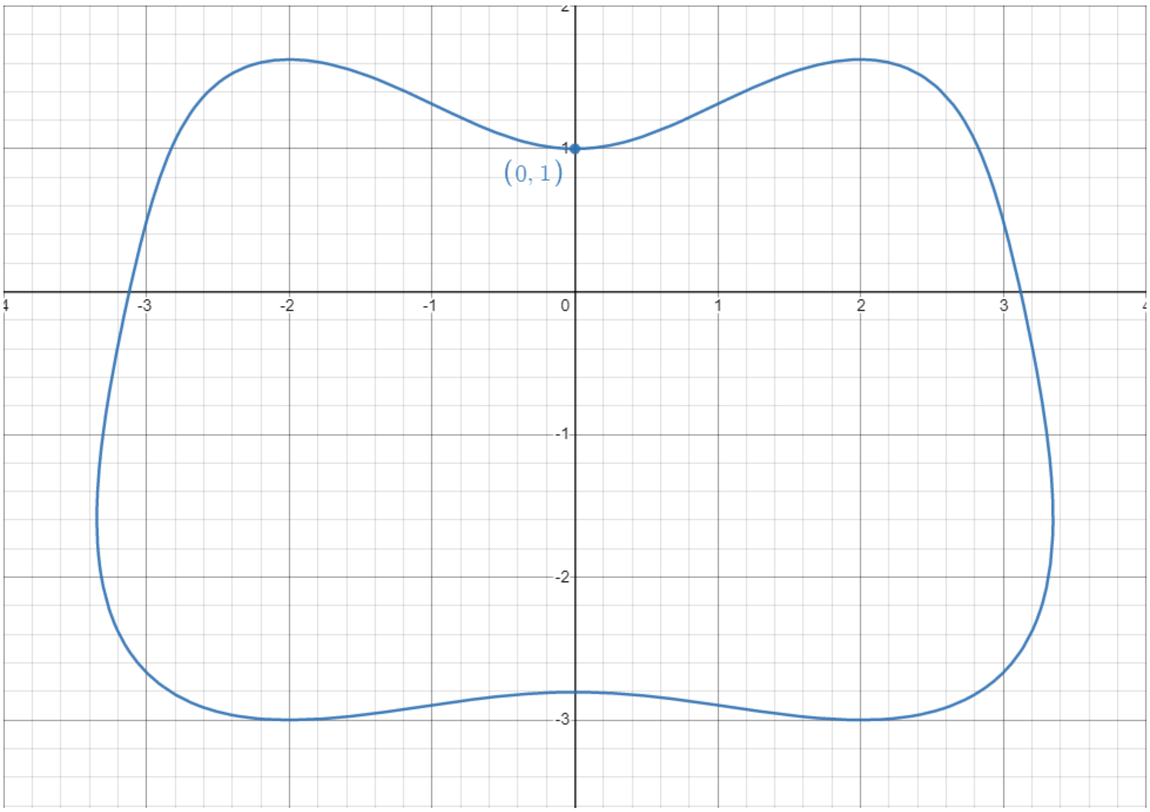
Find the solution of $\frac{dy}{dt} = \frac{4t-t^3}{4+y^3}$ that goes through $(0,1)$

$$\begin{aligned}(4+y^3)dy =&(4t-t^3)dt \\ \int 4+y^3 dy =& \int 4t - t^3 dt \\ 4y + \frac{y^4}{4} =& 2t^2 - \frac{t^4}{4} + C \\ 16y + y^4 =& 8t^2 - t^4 + 4C \\ y^4 + 16y =& -t^4 + 8t^2 + C\end{aligned}$$

Unfortunately here we cannot solve for y in terms of t, and so we leave the solution in **implicit form** as an **integral** curve.

To find C, plug in $t=0$ and $y=1$ to get $1+16 = 0+0+C \Rightarrow C = 17$

So the solution is the curve $y^4 + 16y = -t^4 + 8t^2 + 17$



Example 2: (extra practice)

$$\begin{cases} tdt - ye^{-t}dy = 0 \\ y(0) = 3 \end{cases}$$

$$\begin{aligned} ye^{-t}dy &= tdt \\ ydy &= te^t dt \\ \int ydy &= \int te^t dt \\ \frac{y^2}{2} &= te^t - \int e^t dt = te^t - e^t + C \qquad \text{(Integration by parts)} \\ y^2 &= 2te^t - 2e^t + \underbrace{2C}_{C} \end{aligned}$$

$$y^2 = 2te^t - 2e^t + C \Rightarrow y = \pm\sqrt{2te^t - 2e^t + C}$$

But if $y = -\sqrt{2te^t - 2e^t + C} \leq 0$, then we cannot have $y(0) = 3$ and so $y = \sqrt{2te^t - 2e^t + C}$.

Finally using $y(0) = 3$ we get

$$3 = \sqrt{2(0) - 2 + C} \Rightarrow 3 = \sqrt{-2 + C} \Rightarrow -2 + C = 9 \Rightarrow C = 11$$

Hence $y = \sqrt{2te^t - 2e^t + 11}$

Note: For even more practice, check out the following video:

Video: Separation of Variables

2. Why this works

When solving $\frac{dy}{dt} = \frac{f(t)}{g(y)}$ using separation of variables you get

$$g(y)dy = f(t)dt \Rightarrow G(y) = F(t) + C$$

where F and G are anti-derivatives of f and g

Differentiating this using the chain rule, we get

$$(G(y))' = (F(t) + C)' \Rightarrow G'(y)\frac{dy}{dt} = f(t) \Rightarrow g(y)\frac{dy}{dt} = f(t) \Rightarrow \frac{dy}{dt} = \frac{f(t)}{g(y)} \checkmark$$

So indeed the y we found is a solution of our ODE.

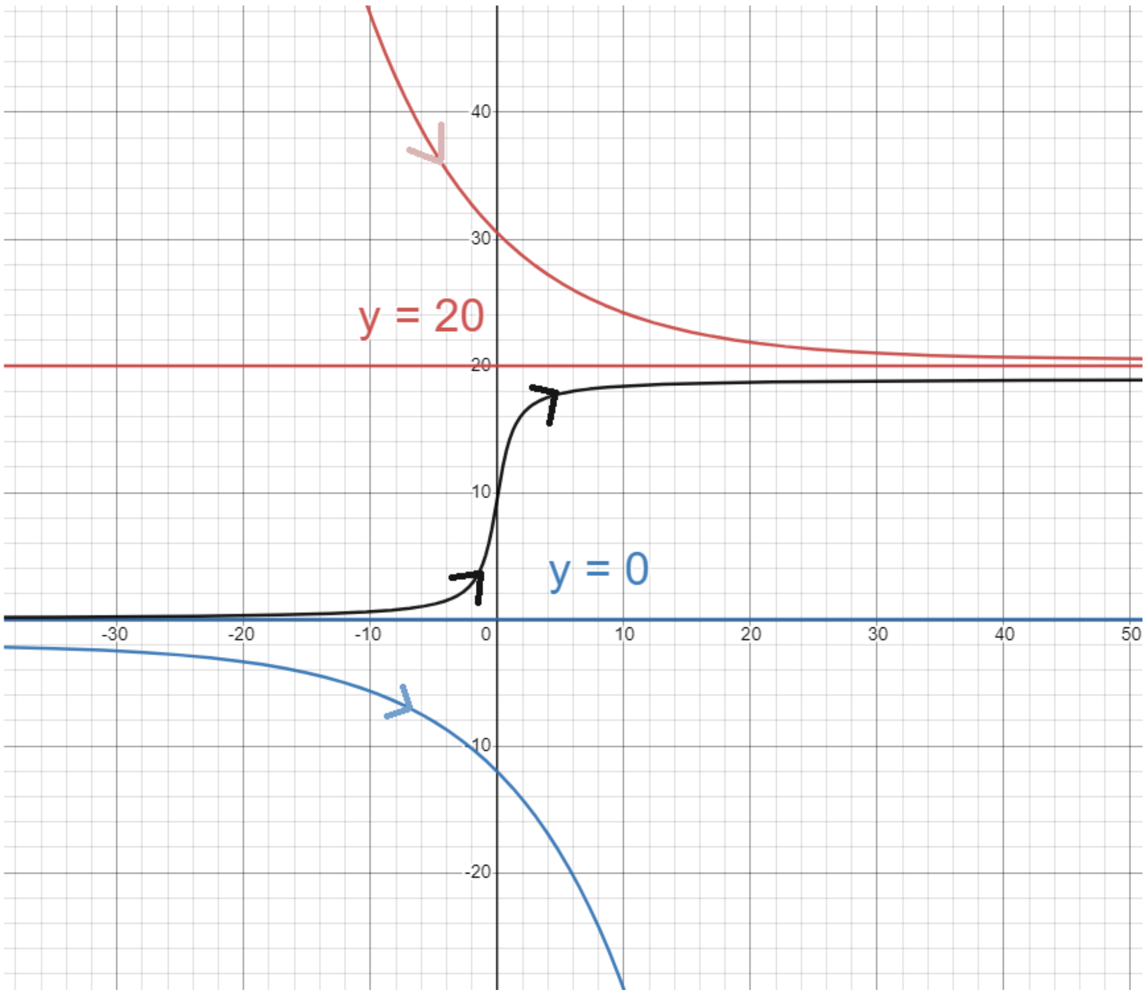
3. Logistic equation

Video: Logistic Equation

Recall: Logistic equation, which models population of bacteria

$$y' = 3y\left(1 - \frac{y}{20}\right)$$

We were able to describe it qualitatively, and got a picture as follows



We now have all the tools we need to solve it

Example 3:

$$\begin{cases} \dfrac{dy}{dt} = 3y\left(1 - \dfrac{y}{20}\right) \\ y(0) = 10 \end{cases}$$

STEP 1:

$$\begin{aligned} dy =& (3y)\left(1-\frac{y}{20}\right) dt \\ \frac{dy}{y\left(1-\frac{y}{20}\right)} =& 3dt \\ \int \frac{dy}{y\left(\frac{20-y}{20}\right)} =& \int 3dt \\ \int \frac{20}{y\,(20-y)} dy =& 3t + C \end{aligned}$$

STEP 2: To evaluate the left hand side, use partial fractions:

Fact:

$$\frac{20}{y\,(20-y)} = \frac{1}{y} + \frac{1}{20-y}$$

Why?

$$\frac{20}{y\,(20-y)} = \frac{A}{y} + \frac{B}{20-y} = \frac{20A - Ay + By}{y\,(20-y)} = \frac{20A + (B-A)y}{y\,(20-y)}$$

$$\begin{cases} 20A =& 20 \\ (B-A) =& 0 \end{cases} \Rightarrow \begin{cases} A =& 1 \\ B =& 1 \end{cases}$$

$$\frac{20}{y\,(20-y)} = \frac{1}{y} + \frac{1}{20-y}$$

STEP 3: Go back to:

$$
\begin{aligned}
\int \frac{20}{y\,(20-y)} dy =& 3t + C \\
\int \frac{1}{y} + \frac{1}{20-y} dy =& 3t + C \\
\ln|y| - \ln|20-y| =& 3t + C \\
\ln\left|\frac{y}{20-y}\right| =& 3t + C \\
\left|\frac{y}{20-y}\right| =& e^{3t+C} \\
\frac{y}{20-y} =& \underbrace{\pm e^{C}}_{C} e^{3t} = Ce^{3t}
\end{aligned}
$$

STEP 4: Cross-multiply

$$
\begin{aligned}
y =& Ce^{3t}\,(20-y) \\
y =& 20Ce^{3t} - Cye^{3t} \\
y + Cye^{3t} =& 20Ce^{3t} \\
y\left(1 + Ce^{3t}\right) =& 20Ce^{3t} \\
y =& \frac{20Ce^{3t}}{1+Ce^{3t}}
\end{aligned}
$$

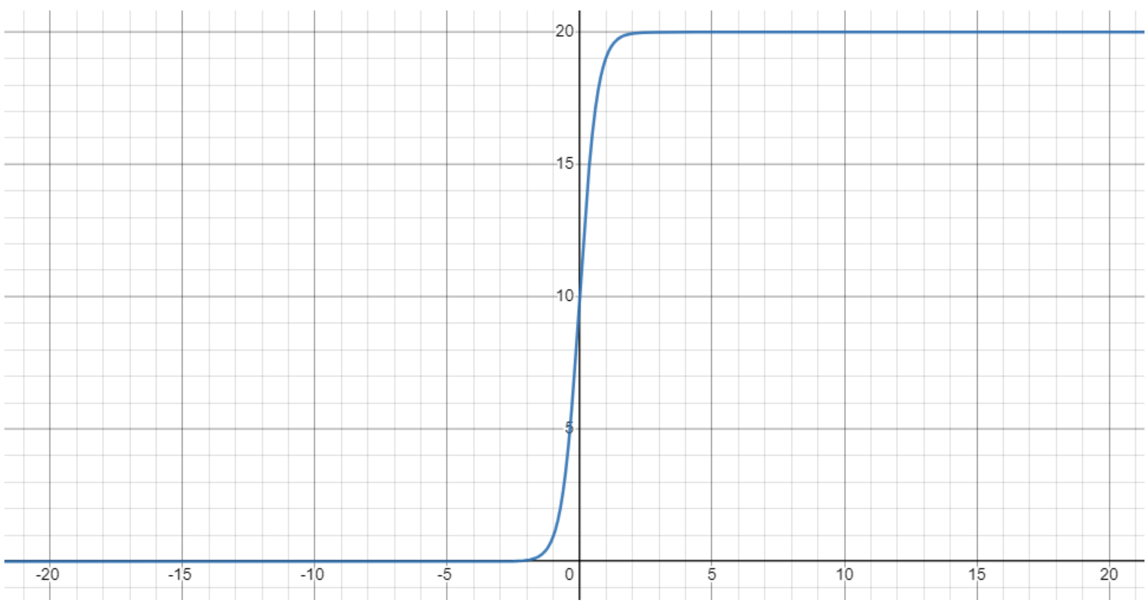
DON'T relabel numerator as C because there is also C on the denom

STEP 5: Finally, to find C, use $y(0) = 10$

$$\begin{aligned} y(0) &= 10 \\ \frac{20C}{1+C} &= 10 \\ 20C &= 10(1+C) \\ 20C &= 10 + 10C \\ 10C &= 10 \\ C &= 1 \end{aligned}$$

STEP 6: Therefore we finally get the solution (the simplification is optional)

$$y = \frac{20e^{3t}}{1+e^{3t}} = \frac{20e^{3t}}{1+e^{3t}}\left(\frac{e^{-3t}}{e^{-3t}}\right) = \frac{20}{1+e^{-3t}}$$



Notice in particular that the behavior is as expected, we start with 10 and then increase until we reach the limit which is $\frac{20}{1+0} = 20$

4. Hidden Solutions

Example 4: (Cautionary Tale)

$$\frac{dy}{dt} = t\left(y^2\right)$$

$$\begin{aligned}
dy =& ty^2 dt \\
\frac{dy}{y^2} =& tdt \qquad \text{(Here we divided by } y^2\text{)} \\
\int \frac{dy}{y^2} =& \int tdt \\
-\frac{1}{y} =& \frac{t^2}{2} + C \\
y =& \frac{1}{-\left(\frac{t^2}{2} + C\right)} \\
y =& \frac{2}{-t^2 - 2C} \\
y =& \frac{2}{-t^2 + C}
\end{aligned}$$

Careful: We missed a solution here! We divided by y^2, which leaves out the possibility of $y = 0$ (dividing by 0) and in fact $y = 0$ is a perfectly valid solution to this equation!

So in fact we should say that the solutions are

$$y = \frac{2}{-t^2 + C} \qquad \text{or } y = 0$$

The moral of the story is: Always beware of hidden solutions, especially when you divide by y

5. Integrating Factors: Motivation

Here is another cool method for solving first-order ODE, based on the product rule

Example 5: (Motivation)

$$e^{3t}\left(y'\right) + \left(3e^{3t}\right) y = t^2$$

Notice the left hand side is precisely the derivative of $e^{3t}y$, that is:

$$\begin{aligned}\left(e^{3t}y\right)' &= t^2 \\ e^{3t}y &= \frac{1}{3}t^3 + C \\ y &= \frac{1}{3}t^3 e^{-3t} + Ce^{-3t}\end{aligned}$$

The idea behind integrating factors is to put equations in the product form above.

6. Integrating Factors

Example 6:

$$y' + 2y = e^t$$

Trick: Multiply by e^{2t} (this is called an **integrating** factor)

$$
\begin{aligned}
e^{2t}\, y' + 2e^{2t}\, y &= e^{2t}\, e^{t} \\
\left(e^{2t} y\right)' &= e^{3t} \\
e^{2t} y &= \frac{1}{3} e^{3t} + C \\
y &= \frac{1}{3}\left(\frac{e^{3t}}{e^{2t}}\right) + \frac{C}{e^{2t}} \\
y &= \frac{1}{3} e^{t} + C e^{-2t}
\end{aligned}
$$

Fact:

For $y' + ay =$ Some function of t multiply by e^{at}

Will see a proof of this later

Example 7:

$$
\begin{cases} 3y' = 6y + t \\ y(0) = 1 \end{cases}
$$

First write as $3y' - 6y = t$

Important:

Make sure the coefficient of y' is 1

Divide both sides by 3 to get $y' - 2y = \frac{t}{3}$ and then multiply by e^{-2t}:

$$
\begin{aligned}
e^{-2t}y' - 2e^{-2t}y =& e^{-2t}\left(\frac{t}{3}\right) \\
\left(e^{-2t}y\right)' =& \frac{1}{3}te^{-2t} \\
e^{-2t}y =& \int \left(\frac{t}{3}\right) e^{-2t} dt \\
=& \left(\frac{t}{3}\right)\left(\frac{e^{-2t}}{-2}\right) - \int \left(\frac{1}{3}\right)\left(\frac{e^{-2t}}{-2}\right) dt \quad \text{(By parts)} \\
=& -\frac{t}{6}e^{-2t} + \frac{1}{6}\int e^{-2t} dt \\
e^{-2t}y =& -\frac{t}{6}e^{-2t} - \frac{1}{12}e^{-2t} + C \\
y =& -\frac{t}{6} - \frac{1}{12} + Ce^{2t}
\end{aligned}
$$

$$
y(0) = 0 - \frac{1}{12} + Ce^{0} = 1 \Rightarrow -\frac{1}{12} + C = 1 \Rightarrow C = 1 + \frac{1}{12} = \frac{13}{12}
$$

$$
y(t) = -\frac{t}{6} - \frac{1}{12} + \frac{13}{12}e^{2t}
$$