LECTURE: INTEGRATING FACTORS

1. More General Integrating Factors

Example 1:

$$y' + \left(3t^2\right) y = 6t^2$$

STEP 1: Integrating Factor

DON'T multiply by e^{3t^2}

Here we have to do something more general:

Integrating Factor:

For $y' + P(t)y = f(t)$ multiply by $e^{\int P(t)dt}$

Mnemonic: Kinda looks like $ye^{\int p}$

$$P(t) = 3t^2 \Rightarrow e^{\int P(t)dt} = e^{\int 3t^2 dt} = e^{t^3}$$

(No constant because we just want *one* integrating factor)

Multiply by e^{t^3} to get:

$$
\begin{aligned}
e^{t^3}y' + 3t^2 e^{t^3} y =& 6t^2 e^{t^3} \\
\left(e^{t^3}y\right)' =& 6t^2 e^{t^3} \qquad \text{(Product Rule)} \\
e^{t^3}y =& \int 6t^2 e^{t^3} dt \qquad (u = t^3, du = 3t^2 dt \Rightarrow 6t^2 dt = 2du) \\
e^{t^3}y =& \int 2e^u du = 2e^u + C = 2e^{t^3} + C \\
y =& 2 + Ce^{-t^3}
\end{aligned}
$$

Remark: For $y' + ay = f(t)$, the integrating factor is

$$e^{\int a dt} = e^{at}$$

Which is the same thing that we had before.

2. Why this works?

Here is why the integrating factor method works:

Suppose our integrating factor is $g(t)$

STEP 1: Start with $y' + Py = f(t)$ and multiply by g to get

$$
\begin{aligned}
g\left(y' + Py\right) =& gf \\
y'g + Pgy =& fg
\end{aligned}
$$

We want to write the left-hand-side as $(gy)' = y'g + yg'$, just like $\left(e^{t^3}y\right)'$ in the example above.

Comparing the two expressions in blue, we get

$$\begin{aligned} \cancel{y'}g + Pgy &= \cancel{y'}g + yg' \\ \Rightarrow \cancel{y}g' &= Pg\cancel{y} \\ \Rightarrow g' &= Pg \end{aligned}$$

Which is just an ODE for g

STEP 2: Solving this, we get:

$$\begin{aligned} \frac{g'}{g} =& P \\ (\ln|g|)' =& P \\ \ln|g| =& \int P dt + C \\ |g| =& e^{\left(\int P\right)+C} = e^{\int P} e^C \\ g =& \underbrace{\pm e^C}_{C} e^{\int P} \\ g =& C e^{\int P} \end{aligned}$$

Since we just need one integrating factor, choose $C = 1$ to get

$$g(t) = e^{\int P}$$

3. More Examples

Video: Integrating Factors

Example 2:

$$\begin{cases} ty' + 2y =& 4t^2 \\ y(1) =& 2 \end{cases}$$

Warning: Again, make sure the coefficient of y' is 1.

STEP 1: Dividing by t, we get:

$$y' + \left(\frac{2}{t}\right) y = 4t$$

STEP 2: Integrating Factor

$$P(t) = \frac{2}{t} \Rightarrow e^{\int P(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = \left(e^{\ln|t|}\right)^2 = t^2$$

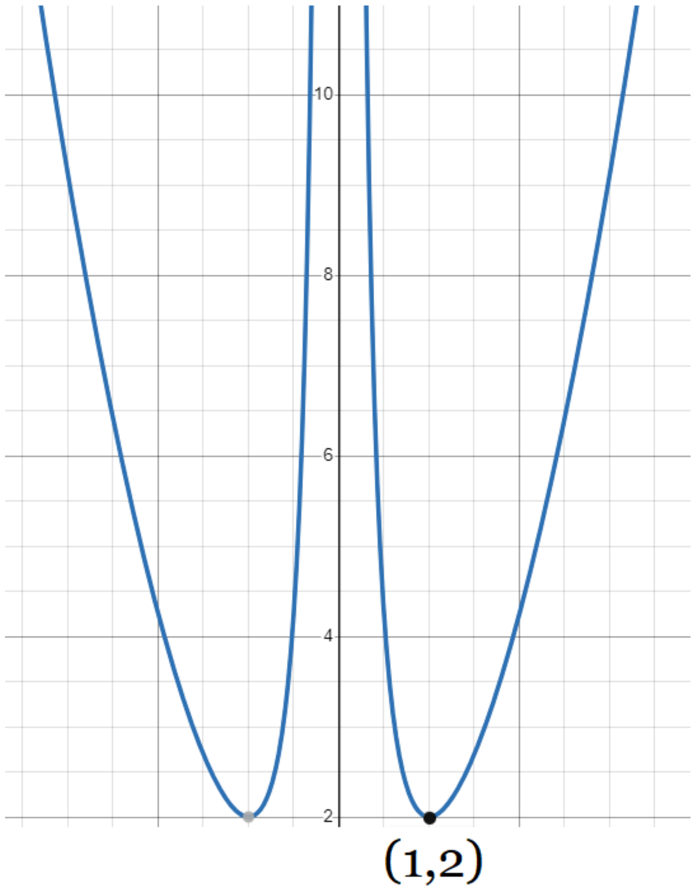
Multiply by t^2 to get:

$$\begin{aligned} t^2\left(y' + \left(\frac{2}{t}\right)y\right) =& t^2\,(4t) \\ \left(t^2 y\right)' =& 4t^3 \\ t^2 y = \int 4t^3 =& t^4 + C \\ y =& t^2 + \frac{C}{t^2} \end{aligned}$$

STEP 3: $y(1) = 2$

$$y(1) = 2 \Rightarrow 1^2 + \frac{C}{1^2} = 2 \Rightarrow 1 + C = 2 \Rightarrow C = 1$$

$$\text{Hence: } y(t) = t^2 + \frac{1}{t^2}$$



Example 3: (extra practice)

$$\begin{cases} y' + \tan(t)y = \cos^2(t) \\ y(0) = -1 \end{cases}$$

STEP 1: Integrating Factor

$$e^{\int P(t)dt} = e^{\int \tan(t)dt} = e^{\ln|\sec(t)|} = \sec(t) = \frac{1}{\cos(t)}$$

STEP 2: Multiply by $\frac{1}{\cos(t)}$ to get:

$$\begin{aligned}
\frac{1}{\cos(t)}y' + \frac{\tan(t)}{\cos(t)}y &= \frac{\cos^2(t)}{\cos(t)} \\
\frac{1}{\cos(t)}y' + \frac{\sin(t)}{\cos^2(t)}y &= \cos(t) \\
\left(\frac{1}{\cos(t)}y\right)' &= \cos(t) \\
\frac{1}{\cos(t)}y &= \int \cos(t)dt = \sin(t) + C \\
y &= \cos(t)\sin(t) + C\cos(t)
\end{aligned}$$

STEP 3: $y(0) = -1$

$$\begin{aligned}
y(0) &= -1 \\
\cos(0)\sin(0) + C\cos(0) &= -1 \\
C &= -1
\end{aligned}$$

Hence $y(t) = \cos(t)\sin(t) - \cos(t)$

Example 4: (extra practice)

$$ty' - 4y = t^6 e^t$$

STEP 1: Dividing by t, we get:

$$y' - \frac{4}{t}y = t^5 e^t$$

STEP 2: Integrating Factor

$$e^{\int P(t)dt} = e^{\int -\frac{4}{t}dt} = e^{-4\ln|t|} = \left(e^{\ln|t|}\right)^{-4} = t^{-4} = \frac{1}{t^4}$$

Multiply by $\frac{1}{t^4}$ to get:

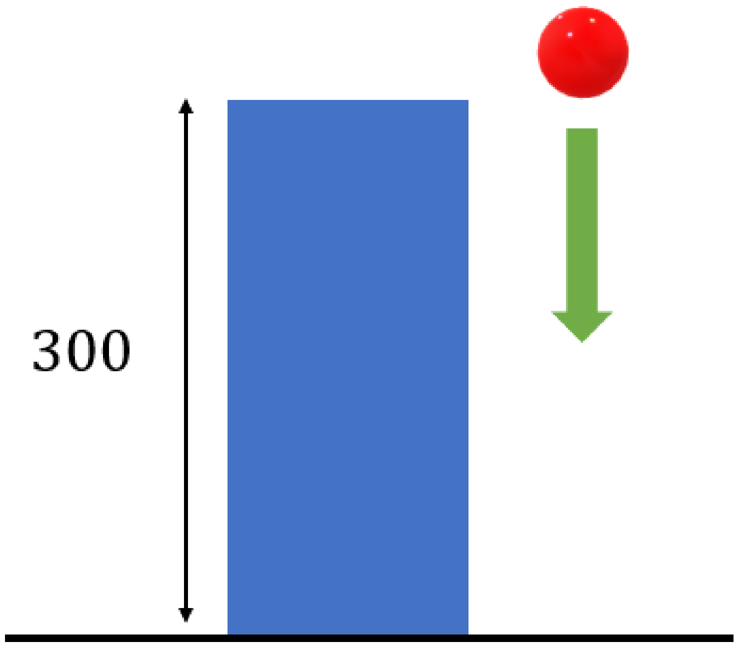
$$\begin{aligned}
\frac{1}{t^4}\left(y' - \frac{4}{t}y\right) =& \frac{1}{t^4}\left(te^t\right) \\
\left(\frac{1}{t^4}\right)y' + \left(\frac{-4}{t^5}\right)y =& te^t \\
\left(\left(\frac{1}{t^4}\right)y\right)' =& te^t \\
\frac{y}{t^4} =& \int te^t dt = te^t - \int e^t dt = te^t - e^t + C \quad \text{(IBP)} \\
y =& t^4\left(te^t - e^t + C\right) \\
y =& t^5e^t - t^4e^t + Ct^4
\end{aligned}$$

4. Example 1: A Falling Object

Consider the following physical scenario:

Example 5:

Suppose an object of mass $m = 10$ kg is dropped from a height of 300 m. What is the velocity $v(t)$ of that object?



STEP 1: Find the ODE

The motion of the object is governed by Newton's Second law:

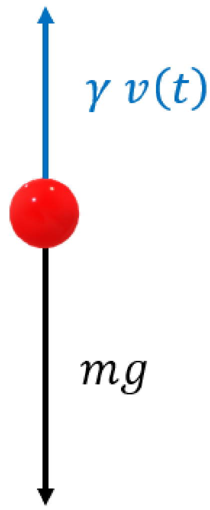
$$F = ma \Rightarrow F(t) = 10a(t)$$

F is the force acting on the object, and $a(t) = v'(t)$ is the acceleration.

$$\text{Hence } F = 10\, v'(t)$$

Study of F:

There are two forces acting on the body: Gravity (pulling down) and Friction (pulling up)



Gravity: mg

Here $g = 9.8m/s^2$ This makes sense, because the heavier the object, the more it gets pulled down

Friction: $-\gamma\, v(t)$

Here $\gamma = 2kg/s$ is called a drag constant, and the $-$ sign is because friction points the opposite direction as gravity.

Combining, we get:

$F = \text{ Gravity } + \text{ Friction } = mg - \gamma v(t) = 9.8(10) - 2v(t) = 98 - 2v(t)$

And using $F = 10v'(t)$, we get:

$$10v'(t) = 98 - 2v(t) \Rightarrow v'(t) = \frac{98 - 2v(t)}{10}$$

Therefore our differential equation becomes:

$$v'(t) = 9.8 - 0.2v(t)$$

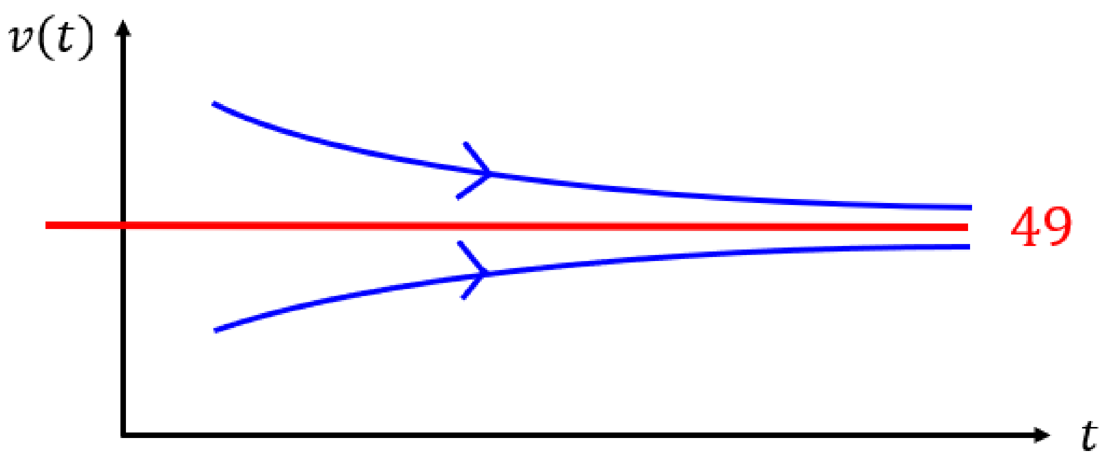
STEP 2: Qualitative Analysis

Equilibrium Solution:

$$9.8 - 0.2v(t) = 0 \Rightarrow v(t) = \frac{9.8}{0.2} = 49$$

Bifurcation Diagram:

v	$-\infty$		49		∞
v'(t)		+	0	−	
v(t)		↗		↘	



Here $v = 49$ is stable, and it seems that $v(t)$ always goes to 49 as $t \to \infty$, no matter what the initial condition.

STEP 3: Solve the ODE

$$v'(t) + 0.2v(t) = 9.8$$

This is of the form $y' + ay = f(t)$ so use integrating factors

Multiply by $e^{at} = e^{0.2t}$

$$\begin{aligned}
e^{0.2t}v'(t) + 0.2e^{0.2t}v(t) =& 9.8e^{0.2t} \\
\left(e^{0.2t}v(t)\right)' =& 9.8e^{0.2t} \\
e^{0.2t}v(t) =& \int 9.8e^{0.2t} = \frac{9.8}{0.2}e^{0.2t} + C \\
e^{0.2t}v(t) =& 49e^{0.2t} + C \\
v(t) =& 49 + \frac{C}{e^{0.2t}}
\end{aligned}$$

$$v(t) = 49 + Ce^{-0.2t}$$

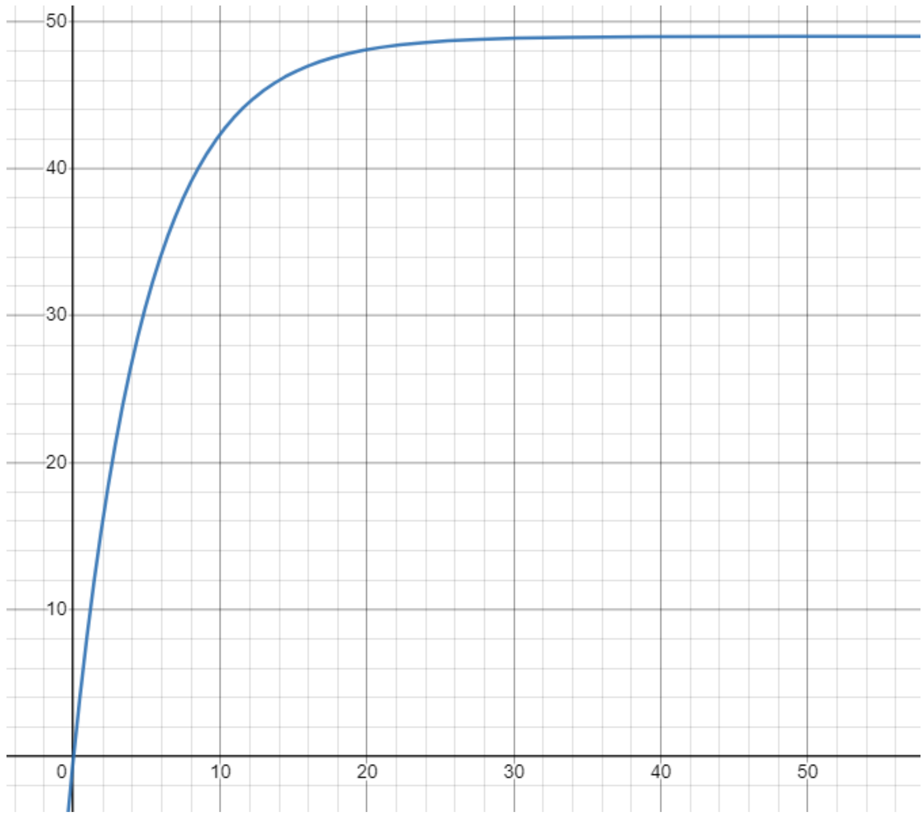
And so indeed, we get that $v(t) \to 49$ as $t \to \infty$. And the amazing fact is that this does not depend on the initial velocity $v(0)$!!

Example 6:

Solve the ODE with $v(0) = 0$

$$v(0) = 49 + Ce^{0} = 49 + C = 0 \Rightarrow C = -49$$

$$v(t) = 49 - 49e^{-0.2t}$$



Example 7:

Find the position $x(t)$ of the particle

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int 49 - 49e^{-0.2t}dt \\ &= 49t - \left(\frac{49}{-0.2}\right)e^{-0.2t} + C \\ &= 49t + 245e^{-0.2t} + C \end{aligned}$$

To find C, use $x(0) = 0$

Note: This initial condition is open to interpretation, you could require $x(0) = 300$ but keep in mind that the axes are pointing downwards, so if you assume this, below you need to solve $x(t) = 600$ below.

$$\begin{aligned} x(0) =& 0 \\ 49(0) + 245e^0 + C =& 0 \\ 245 + C =& 0 \\ C =& -245 \end{aligned}$$

$$x(t) = 49t + 245e^{-0.2t} - 245$$

Note: It *looks* like $x(t)$ is increasing, but remember that our axes are pointing down, so the ball is indeed falling

And from this you can answer things like "At what time does the object hit the ground?" by solving $x(t) = 300$, which has to be done numerically here