LECTURE: APPLICATIONS

Today: More applications of first-order ODE

1. Example 2: To the Moon!

Example 2:

Suppose you put \$100 in a savings account that pays interest at an annual rate of 5% (compounded continuously)

How long will it take for the money to reach \$250?

Let $S(t)$ be the value of the savings at time t.

STEP 1: Derivation

Let h be a small change in time, think $h = 1$ second

In h amount of time, the savings will increase by $(0.05S(t))\,h$ dollars

$$\text{Change} = S(t+h) - S(t) = 0.05S(t)h \Rightarrow \frac{S(t+h) - S(t)}{h} = 0.05S(t)$$

Taking the limit as $h \to 0$ we get

$$S'(t) = 0.05\,S(t)$$

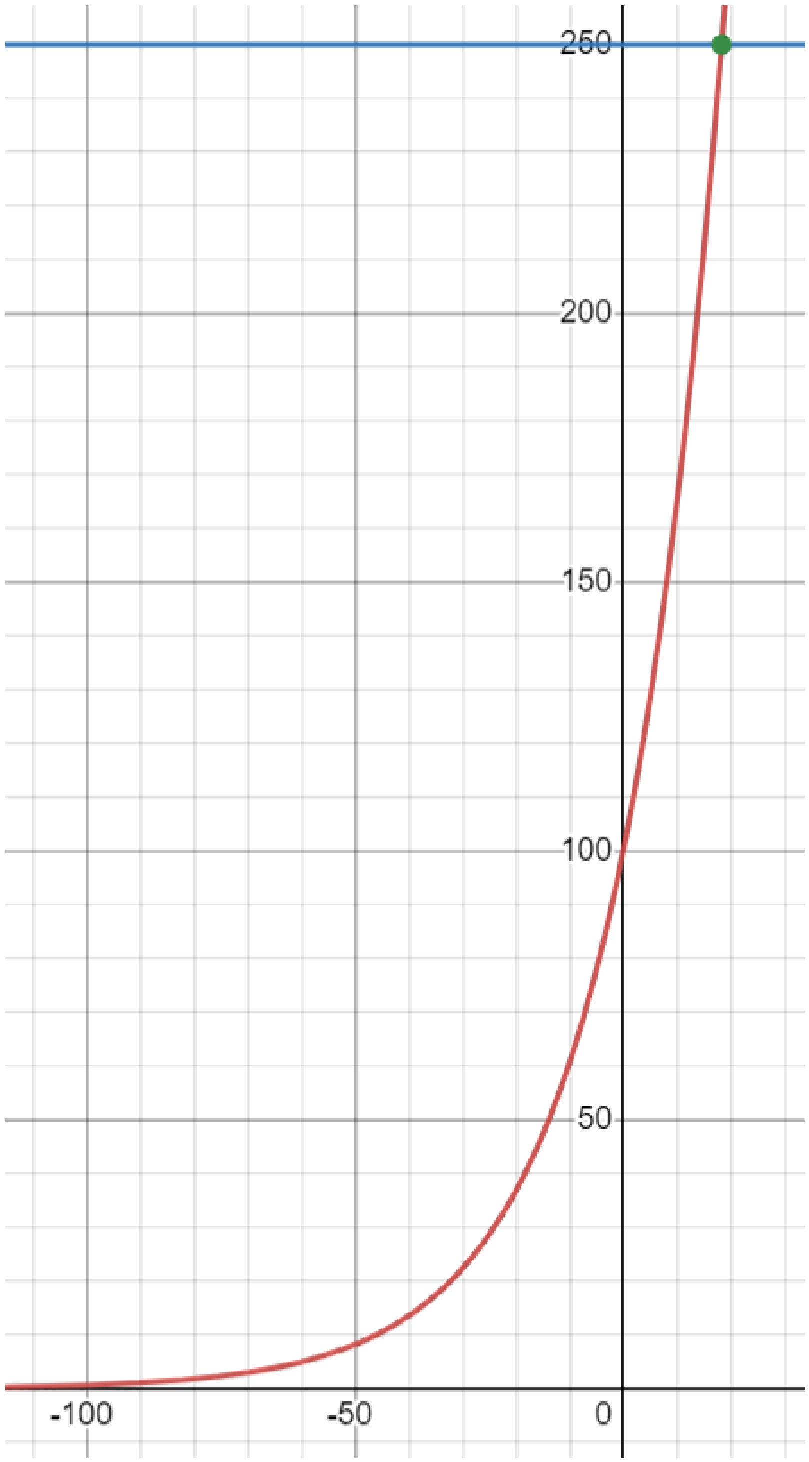
STEP 2: Solution

This is just our basic ODE, and so $S(t) = Ce^{0.05t}$

$$S(0) = 100 \Rightarrow C = 100 \Rightarrow S(t) = 100e^{0.05t}$$

STEP 3:

$$\begin{aligned} S(t) &= 250 \\ 100e^{0.05t} &= 250 \\ e^{0.05t} &= 2.5 \\ 0.05t &= \ln(2.5) \\ t &= \frac{\ln(2.5)}{0.05} \\ t &\approx 18.3 \text{ years} \end{aligned}$$


250
200
150
100
50
-100
-50
0

Which is kind of crazy if you think about it! You'd have to wait 2 decades for your money to reach \$250. There is nothing wrong with this model, and 5% is pretty generous actually. It shows that savings isn't always the best investment.

Example:

Same question but this time you also contribute \$10 a year (compounded continuously)

STEP 1: Derivation:

This time you get $0.05S(t)h$ dollars but also contribute $10h$ and so

$$S(t+h) - S(t) = 0.05S(t)h + 10h \Rightarrow \frac{S(t+h) - S(t)}{h} = 0.05S(t) + 10$$

Taking the limit as $h \to 0$ we get the ODE

$$S'(t) = (0.05)S(t) + 10 = \text{ Savings growth } + \text{ Contributions}$$

STEP 2: Solution:

$S'(t) - 0.05S(t) = 10 \Rightarrow$ Use Integrating Factors

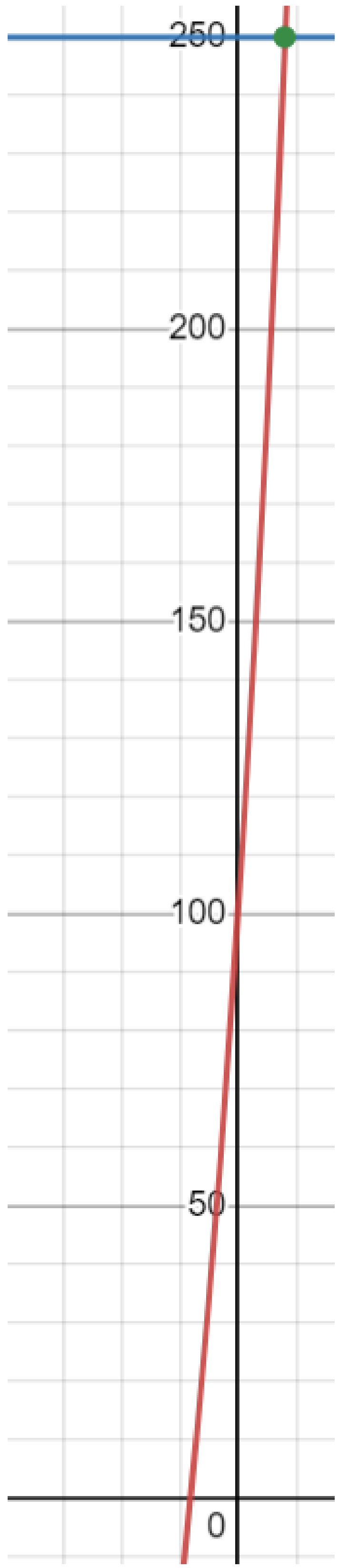
Multiplying by $e^{-0.05t}$ we get:

$$\begin{aligned}
e^{-0.05t}S' - 0.05e^{-0.05t}S =& 10e^{-0.05t} \\
\left(e^{-0.05t}S\right)' =& 10e^{-0.05t} \\
e^{-0.05t}S =& \int 10e^{-0.05t}dt = \frac{10e^{-0.05t}}{-0.05} + C \\
e^{-0.05t}S =& -200\, e^{-0.05t} + C \\
S(t) =& -200 + Ce^{0.05t}
\end{aligned}$$

$$S(0) = 100 \Rightarrow -200 + Ce^{0} = 100 \Rightarrow C = 300$$

$$S(t) = 300e^{0.05t} - 200$$

$$\begin{aligned}
S(t) =& 250 \\
300e^{0.05t} - 200 =& 250 \\
300e^{0.05t} =& 450 \\
e^{0.05t} =& \frac{450}{300} \\
0.05t =& \ln(1.5) \\
t =& \frac{\ln(1.5)}{0.05} \\
t \approx& 8 \text{ years}
\end{aligned}$$


250
200
150
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0

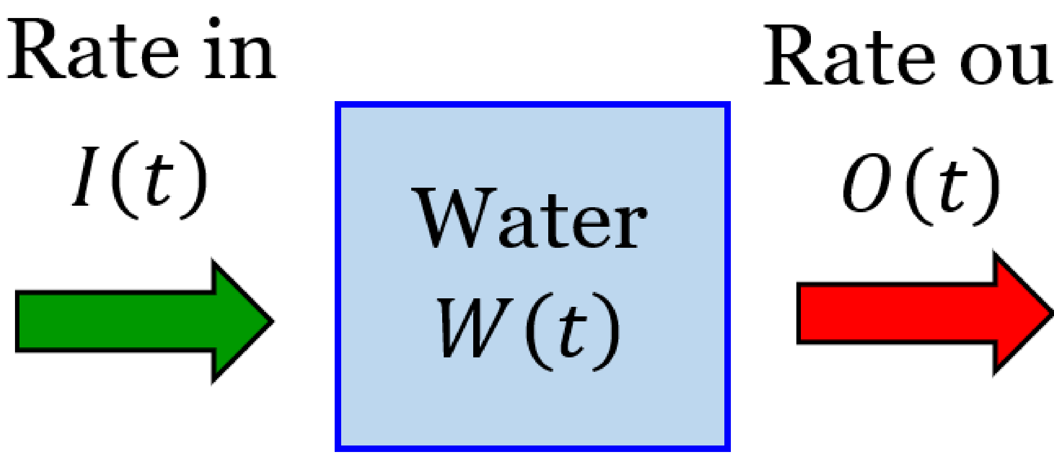
This is a bit more reasonable! That said, it still shows that savings accounts aren't necessarily the best, because here you contributed about $100 + 8 \times 10 = 180$, so your actual gain over the 8 years is \$70.

2. Interlude: Compartmental Models

To prepare for the next example, let's discuss an important class of examples called a compartmental model.

Example:

Suppose water flows into a bathtub at a rate of $I(t)$ gallons/s (inflow rate) and flows out at a rate of $O(t)$ gallons/s (outflow rate). Find a differential equation for the amount of water $W(t)$



Main Idea: Again, pick a small time increment h, then

$$\begin{aligned}
\text{Change } &= W(t+h) - W(t) \\
&= \text{ Rate in } \times \text{ Elapsed Time } - \text{ Rate out } \times \text{ Elapsed time} \\
&= I(t)h - O(t)h \\
&= h\left(I(t) - O(t)\right)
\end{aligned}$$

$$\text{This gives: } \frac{W(t+h) - W(t)}{h} = I(t) - O(h)$$

Taking the limit as $h \to 0$, we then get the ODE

$$W'(t) = I(t) - O(t)$$

In other words:

$$\text{Rate of change} \;=\; \text{Amount entering} - \text{Amount exiting (per unit time)}$$

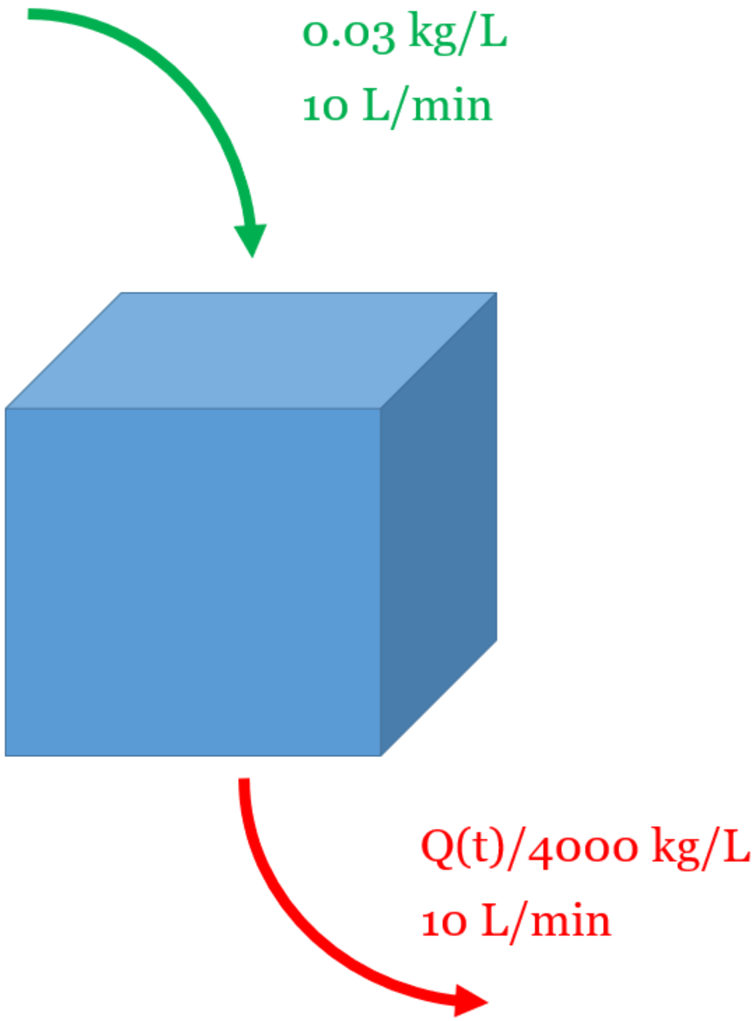
3. Example 3: Chemical Reactions

Example 3:

A tank contains 20 kg of salt dissolved in 4000 L of water.

Water containing 0.03 kg/L of salt is entering the tank at a rate of 10 L/min and the mixture is draining from the tank at the same rate.

How much salt remains in the tank after half an hour?



STEP 1: Let $Q(t)$ be the amount of salt after t minutes in kg.

For example $Q(0) = 20$ kg

STEP 2: Differential Equation:

There are two forces in play here, the mixture flowing in and the mixture leaking out, so we have:

$$\frac{dQ}{dt} = \text{ Rate in } - \text{ Rate out}$$

Rate in: The amount of salt being pumped in, measured in kg/min:

$$\underbrace{\text{Rate in}}_{\text{kg/min}} = \underbrace{\text{Concentration}}_{\text{kg/L}} \times \underbrace{\text{Rate}}_{\text{L/min}} = \ (0.03 \text{ kg/L}) \times (10 \text{ L/min}) \ = 0.3 \text{ kg/min}$$

Rate out: The amount of salt being pumped out.

The concentration of salt in the tank is $\frac{\text{Weight}}{\text{Volume}} = \frac{Q(t)}{4000}$, so:

$$\text{Rate out} \ = \ \text{Concentration} \ \times \ \text{Rate} \ = \frac{Q(t)}{4000} \times 10 = \frac{Q(t)}{400} \text{ kg/min}$$

Therefore our differential equation is:

$$\frac{dQ}{dt} = 0.3 - \frac{Q(t)}{400} \Rightarrow Q'(t) + \frac{Q(t)}{400} = 0.3$$

Initial Condition: $Q(0) = 20$

STEP 3: Solve the ODE

Multiply by $e^{\frac{t}{400}}$

$$\begin{aligned} e^{\frac{t}{400}}Q' + \left(\frac{1}{400}\right)e^{\frac{t}{400}}Q =& 0.3e^{\frac{t}{400}} \\ \left(e^{\frac{t}{400}}Q\right)' =& 0.3e^{\frac{t}{400}} \\ e^{\frac{t}{400}}Q =& \int 0.3e^{\frac{t}{400}}dt \\ e^{\frac{t}{400}}Q =& \frac{0.3}{\frac{1}{400}}e^{\frac{t}{400}} + C = 120e^{\frac{t}{400}} + C \\ Q =& 120 + Ce^{-\frac{t}{400}} \end{aligned}$$

$$Q(0) = 20 \Rightarrow 120 + C = 20 \Rightarrow C = -100$$

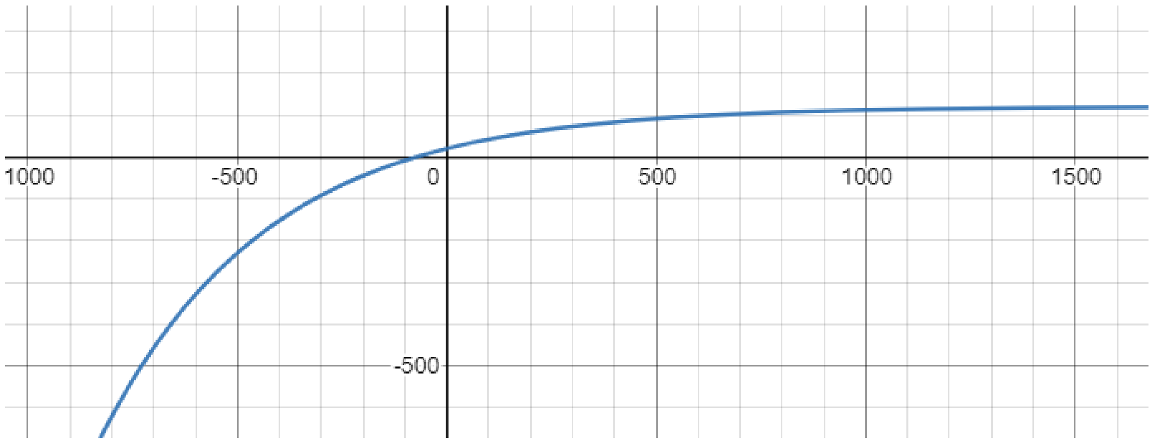
$$Q(t) = 120 - 100e^{-\frac{t}{400}}$$

STEP 4: Answer

Half an hour corresponds to 30 mins, so

$$Q(30) = 120 - 100e^{-\frac{30}{400}} \approx 27.22 \text{ kg}$$

Note: In the long run, as $t \to \infty$, the amount of salt becomes 120 kg, which is precisely 4000 L $\times 0.03$ kg/L (volume of tank times rate flowing in)



Note: Another application is Newton's Law of Cooling, which will be discussed on the homework

4. Example 4: Bunnies and Foxes

Note: I will not cover this example in class since it's similar to ones we've studied, but I still recommend you to look at it, especially the derivation.

Let's now study another population model sometimes called a **predator-prey** model[1]

[1] The picture is from my bunny Oreo (2014-2021)

Example 4:

Suppose that, on average, a bunny population reproduces at a rate of 0.5 bunnies a month, but gets invaded by a population of foxes that eat 450 bunnies per month. Find the population $P(t)$ of bunnies

Note: If no foxes are present, the bunny population is governed by

$$P'(t) = 0.5P(t)$$

STEP 1: ODE for $P(t)$

Let h be a small time increment

In h months, $0.5h\,P(t)$ bunnies get produced and $450h$ bunnies get eaten (doesn't depend on number of bunnies present), so

$$\begin{aligned}\text{Change } &= P(t+h) - P(t) = 0.5hP(t) - 450h \\ \frac{P(t+h) - P(t)}{h} &= 0.5P(t) - 450\end{aligned}$$

Taking the limit as $h \to 0$, we get

$$P'(t) = 0.5P(t) - 450$$

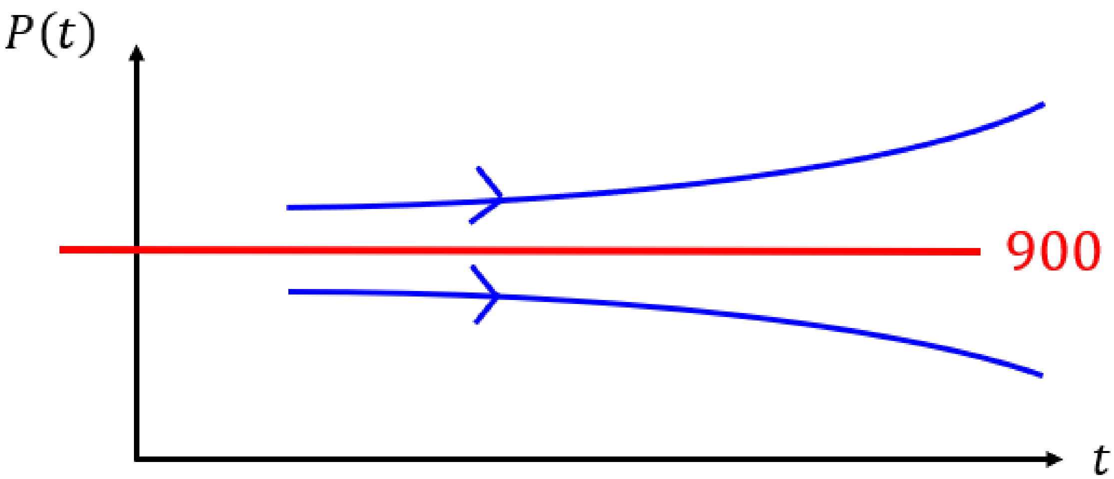
STEP 2: Qualitative Analysis

Equilibrium Solution:

$$0.5P(t) - 450 = 0 \Rightarrow P(t) = \frac{450}{0.5} = 900$$

Bifurcation Diagram:

P	0		900		∞
P'(t)		$-$	0	$+$	
P(t)		$\searrow$	$\vert$	$\nearrow$	



Here the equilibrium solution $P = 900$ is unstable

The interpretation of this is super interesting:

(1) If the initial bunny population is over 900, then the foxes have pretty much no effect on the population, in the sense that $P(t)$ will increase and blow up to ∞

(2) If it is under 900, then the foxes will eventually kill off the bunny population, $P(t)$ will go to 0 as $t \to \infty$

(3) If the bunny population is 900, then it will always be 900. In that case, the foxes *precisely* balances out the bunny population to keep it constant

STEP 3: Solve the ODE

$$P'(t) - 0.5P(t) = -450$$

Once again use the integrating factor $e^{-0.5t}$:

$$\begin{aligned}
e^{-0.5t}P'(t) - 0.5e^{-0.5t}P(t) &= -\,450e^{-0.5t} \\
\left(e^{-0.5t}P(t)\right)' &= -\,450e^{-0.5t} \\
e^{-0.5t}P(t) &= \int -450e^{-0.5t} \\
e^{-0.5t}P(t) &= \frac{-450}{-0.5}e^{-0.5t} + C = 900e^{-0.5t} + C \\
P(t) &= 900 + \frac{C}{e^{-0.5t}} \\
P(t) &= 900 + Ce^{0.5t}
\end{aligned}$$

Example:

Find $P(t)$ in the following three cases:

(a) $P(0) = 850$

(b) $P(0) = 950$

(c) $P(0) = 900$

(a)

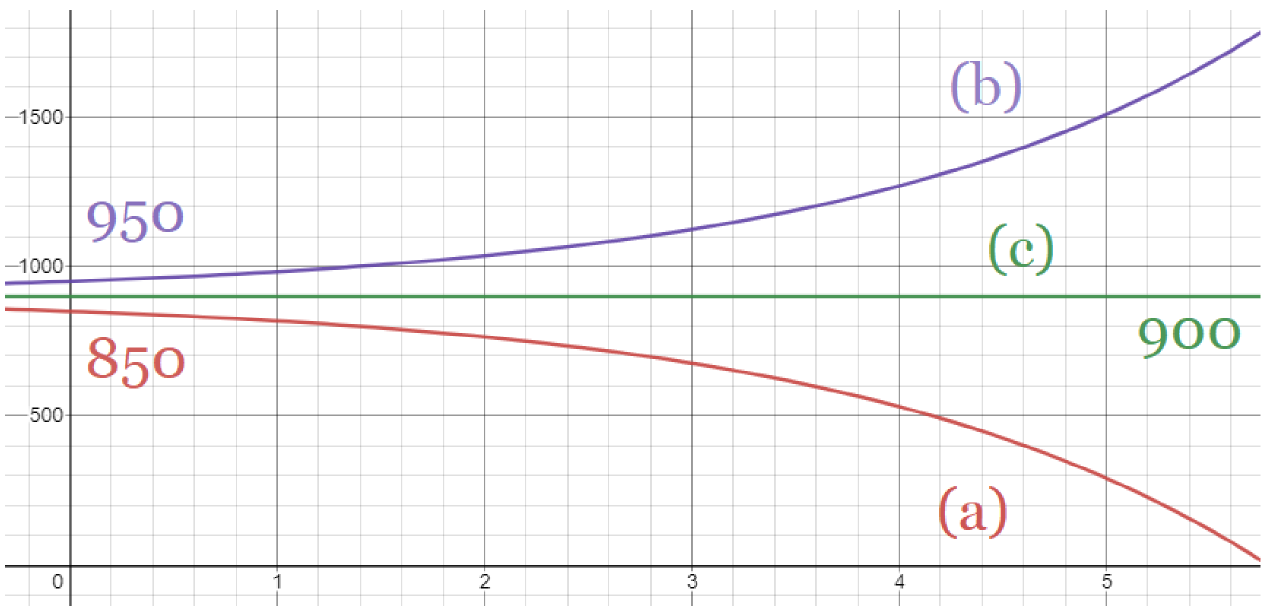
$$P(0) = 850 \Rightarrow 900 + Ce^{0.5(0)} = 850 \Rightarrow 900 + C = 850 \Rightarrow C = -50$$

And therefore we get $P(t) = 900 - 50e^{0.5t}$

Notice that P decreases until it eventually reaches 0, in which case the bunny population gets wiped out.

(b) Similarly, in that case we get $C = 50$ and so $P(t) = 900 + 50e^{0.5t}$. In that case the bunny population grows without bound, almost as if there are no foxes

(c) In that case, we surprisingly get $C = 0$ and so $P(t) = 900$, which means that the foxes eat enough bunnies for the population to not change at all!



Is this an accurate model for population growth? Probably not, the bunny population doesn't just shoot off to ∞ or $-\infty$. This is why we need better models, like the logistic equation $y' = 3y\left(1 - \frac{y}{20}\right)$ that we discussed before.