LINEAR ALGEBRA REVIEW

Welcome to our quick excursion into the world of linear algebra, which is the study of vectors, matrices, and linear equations.

1. Vectors and Matrices

Definition: (Vector)

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 1: (Dot Product)

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 8 \end{bmatrix} = (2)(5) + (4)(8) = 10 + 32 = 42$$

Definition: (Matrix)

$$A = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$$

(just a 2×2 table of numbers)

Note: In this course we'll mainly study 2×2 matrices.

Date: Friday, March 24, 2023.

Example 2: (Addition and Scalar Multiplication)

$$\begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ -2 & -7 \end{bmatrix} = \begin{bmatrix} 1+3 & 2+5 \\ 4-2 & 6-7 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 2 & -1 \end{bmatrix}$$

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Can also multiply a matrix with a vector, which is just a dot product:

Example 3:

$$\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (1)(-1)+(2)(2) \\ (5)(-1)+(2)(3) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

You just take the dot product of each **row** of A with the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example 4:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1)x_1 + (2)x_2 \\ (3)x_1 + (4)x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

This explains the process of "writing systems in matrix form"

Note: If any of this piques your interest, in the video below presents a non-technical overview of some of the main concepts in linear algebra:

Video: Linear Algebra Overview

2. Matrix Multiplication

Video: Matrix Multiplication

The process for multiplying two **matrices** is similar, but trickier:

Example 5:

Calculate AB where

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$

In other words, we would like to calculate

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$

STEP 1: Start with the first **row** of A and first **column** of B and take a dot product:

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} (1)(2) + (2)(0) & \star \\ \star & \star \end{bmatrix} = \begin{bmatrix} 2 & \star \\ \star & \star \end{bmatrix}$$

STEP 2: Fix the first row, but move on to the **second** column of B:

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & (1)(3) + (2)(4) \\ \star & \star \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ \star & \star \end{bmatrix}$$

STEP 3: We ran out of columns, so now dot the **second** row of A and the first column of B:

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ (-1)(2) + (1)(0) & \star \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ -2 & \star \end{bmatrix}$$

STEP 4: Finally, dot the **second** of A and the second column of B

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ -2 & (-1)(3) + (1)(4) \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ -2 & 1 \end{bmatrix}$$

Example 6:

Calculate AB and BA where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -3 \\ 0 & -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} (1)(1) + (2)(0) & (1)(-3) + (2)(-4) \\ (3)(1) + (4)(0) & (3)(-3) + (4)(-4) \end{bmatrix} = \begin{bmatrix} 1 & -11 \\ 3 & -25 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -3 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (1)(1) + (-3)(3) & (1)(2) + (-3)(4) \\ (0)(1) + (-4)(3) & (0)(2) + (-4)(4) \end{bmatrix} = \begin{bmatrix} -8 & -10 \\ -12 & -16 \end{bmatrix}$$

Warning:

In general, $AB \neq BA$

Basically, matrices are weird.

As another example, $AB = AC \not\Rightarrow B = C$

Sidenote: You have already seen an instance of this in calculus, where $f(g(x)) \neq g(f(x))$, putting your socks on and then your shoes is not the same as putting your shoes on and then your socks.

Definition: (Identity Matrix)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Fact:

$$AI = IA = A \quad \text{for any matrix } A$$

I is the analog of the number 1 in the matrix world, since for any real number x, we have $1x = x1 = x$

3. Matrix Inverses

Finally, we can calculate the inverse A^{-1} of a matrix A

Warning: This trick only works for 2×2 matrices, do **NOT** attempt for bigger matrices (but see this video if you're curious about the higher-order case)

Definition: (2×2 Inverse)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

You swap the diagonal entries and you negate the other entries

Example 7:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{(1)(4)-(2)(3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Example 8:

$$\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{\underbrace{(2)(4)-(7)(1)}_{1}} \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

Just like for real numbers, we have $x\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)x = 1$, here have:

Fact:

$$AA^{-1} = A^{-1}A = I$$

So A^{-1} literally "undoes" whatever A does, like a "cancel" button.

Fun Fact:

$$(AB)^{-1} = B^{-1}A^{-1}$$

Interpretation: To undo "putting your socks on and then your shoes," you first remove your *shoes* and then you remove your *socks*, in the reverse order.

4. Gaussian Elimination

Video: Gaussian Elimination

Welcome to the holy grail of linear algebra: Gaussian elimination. It's a tool that allows us to easily solve systems of equations.

Example 9:

$$\begin{cases} x + 3y = 7 \\ 2x - 5y = -8 \end{cases}$$

STEP 1: Write in matrix form

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -5 & -8 \end{array}\right]$$

STEP 2: Use elementary row operations (EROS) to transform the matrix into triangular form

Allowable moves:

(1) Interchange two rows

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

(2) Multiply one row by a nonzero number

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{2R_1} \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$$

(3) Most common: Add/Subtract a multiple of one row to another

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 + (3R_1)} \begin{bmatrix} 1 & 2 \\ 3 + 3 \times 1 & 4 + 3 \times 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 6 & 10 \end{bmatrix}$$

Goal: Transform the system in triangular form:

$$\left[\begin{array}{cc|c} \star & \star & \star \\ 0 & \star & \star \end{array}\right]$$

In our system, this becomes:

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -5 & -8 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5-2\times 3 & -8-2\times 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -11 & -22 \end{array}\right]$$

STEP 3: Backsubstitution

Use EROS to transform the system into the form

$$\left[\begin{array}{cc|c} 1 & 0 & \star \\ 0 & 1 & \star \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -11 & -22 \end{array}\right] \xrightarrow{\div(-11) R_2} \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 3-3 & 7-6 \\ 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

In terms of variables, what this tells us is

$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$

Example 10:

$$\begin{cases} 2x + 3y = -5 \\ 3x - y = 9 \end{cases}$$

STEP 1:

$$\left[\begin{array}{cc|c} 2 & 3 & -5 \\ 3 & -1 & 9 \end{array}\right]$$

STEP 2: We want to turn that bottom 3 into 0:

$$\left[\begin{array}{cc|c} 2 & 3 & -5 \\ 3 & -1 & 9 \end{array}\right] \xrightarrow{R_2 - \frac{3}{2}R_1} \left[\begin{array}{cc|c} 2 & 3 & -5 \\ \left(-\frac{3}{2}\right)(2) + 3 & \left(-\frac{3}{2}\right)(3) - 1 & \left(-\frac{3}{2}\right)(-5) + 9 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 2 & 3 & -5 \\ 0 & -\frac{11}{2} & \frac{33}{2} \end{array}\right]$$

STEP 3: Backsubstitution

$$\left[\begin{array}{cc|c} 2 & 3 & -5 \\ 0 & -\frac{11}{2} & \frac{33}{2} \end{array}\right] \xrightarrow{2R_2} \left[\begin{array}{cc|c} 2 & 3 & -5 \\ 0 & -11 & 33 \end{array}\right]$$

$$\xrightarrow{(\div -11)R_2} \left[\begin{array}{cc|c} 2 & 3 & -5 \\ 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 2 & 3-3 & -5-3(-3) \\ 0 & 1 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{(\div 2)R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

$$\begin{cases} x = 2 \\ y = -3 \end{cases} \Rightarrow \mathbf{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Here $\mathbf{x}$ is the solution, but in "vector form"

5. Infinitely Many Solutions

Video: Infinitely Many Solutions

Example 11:

$$\begin{cases} 2x + 4y = 8 \\ 3x + 6y = 12 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 4 & 8 \\ 3 & 6 & 12 \end{array}\right] \xrightarrow{(\div 2)R_1\ (\div 3)R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 2 & 4 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

OH NO!! There's no way that we can put it in the form $\left[\begin{array}{cc|c} 1 & 0 & \star \\ 0 & 1 & \star \end{array}\right]$

Here there are infinitely many solutions. Rewriting with x and y:

We get: $x + 2y = 4 \Rightarrow x = 4 - 2y$ and so

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 - 2y \\ y \end{bmatrix} = \begin{bmatrix} 4 - 2y \\ 0 + y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Point: We have infinitely many solutions, one for each y. The graph of the solutions is a line through $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ and with direction vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

6. Eigenvalues and Eigenvectors

Video: Eigenvalues and Eigenvectors

Example 12: (Motivation)

Consider $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (6)(1) \\ (5)(1) + (2)(1) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \mathbf{v}$$

$A\mathbf{v}$ isn't just random, but in fact a multiple of $\mathbf{v}$. In this case, we call $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector of A and $\lambda = 7$ (the multiple) an eigenvalue:

Definition:

If $A\mathbf{v} = \lambda \mathbf{v}$, then:

λ is called an **eigenvalue** of A

$\mathbf{v}$ is an **eigenvector** of A corresponding to λ

Interpr.: If $\mathbf{v}$ is an eigenvector, then $\mathbf{v}$ and $A\mathbf{v}$ lie on the same line!

7. Finding Eigenvalues

Question: How to find eigenvalues?

Example 13:

Find the eigenvalues of $A = \begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix}$

Motivation: This calculation won't really make sense unless you've taken linear algebra, but is a way of remembering the formula.

$$\begin{aligned} &A\mathbf{v} = \lambda \mathbf{v} \\ \Rightarrow &A\mathbf{v} - \lambda \mathbf{v} = \mathbf{0} \\ \Rightarrow &A\mathbf{v} - \lambda I \mathbf{v} = \mathbf{0} \\ \Rightarrow &(A - \lambda I)\mathbf{v} = \mathbf{0} \\ \Rightarrow &\det(A - \lambda I) = 0 \end{aligned}$$

Here $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix and $|A - \lambda I|$ is the determinant.

Mnemonic: $A - \lambda I$ looks like Ali (as in Muhammad Ali)

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{bmatrix} 0 & 6 \\ -1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 0 - \lambda & 6 - 0 \\ -1 - 0 & 5 - \lambda \end{bmatrix}\right) \\
&= \begin{vmatrix} -\lambda & 6 \\ -1 & 5 - \lambda \end{vmatrix} \\
&= (-\lambda)(5 - \lambda) - 6(-1) \\
&= -5\lambda + \lambda^2 + 6 \\
&= \lambda^2 - 5\lambda + 6 \\
&= (\lambda - 2)(\lambda - 3) \\
&= 0
\end{aligned}$$

Which gives $\boxed{\lambda = 2 \text{ or } \lambda = 3}$

Example 14:

Find the eigenvalues of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 6 \\ 5 & 2 - \lambda \end{vmatrix} \qquad \text{(Subtract } \lambda \text{ from the diagonals)} \\
&= (1 - \lambda)(2 - \lambda) - 5(6) \\
&= 2 - \lambda - 2\lambda + \lambda^2 - 30 \\
&= \lambda^2 - 3\lambda - 28 \\
&= (\lambda - 7)(\lambda + 4) \\
&= 0
\end{aligned}$$

$\lambda = 7$ or $\lambda = -4$

8. Finding Eigenvectors

Question: Now how do we find eigenvectors?

Example 15:

Find the eigenvectors of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

STEP 1: Find the eigenvalues: $\lambda = 7$ and $\lambda = -4$

STEP 2: $\lambda = 7$

Motivation: $A\mathbf{v} = \lambda \mathbf{v} \Rightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$

Strategy: For every λ you found, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Note: This is sometimes called the nullspace, Nul $(A - \lambda I)$

$$\begin{aligned} \text{Nul } (A - 7I) &= \left[\begin{array}{cc|c} 1-7 & 6 & 0 \\ 5 & 2-7 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right] \\ &\xrightarrow{(\div -6)R_1 \ (\div 5)R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

This says $x - y = 0$ and so $x = y$ and

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = 7$

Important: You should **never** find $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$!! If you do, you either found the wrong eigenvalue, or you made a mistake in your row reduction!

STEP 3: $\lambda = -4$

$$\begin{aligned} \text{Nul } (A - (-4)I) &= \left[\begin{array}{cc|c} 1 - (-4) & 6 & 0 \\ 5 & 2 - (-4) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 5 & 6 & 0 \\ 5 & 6 & 0 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{cc|c} 5 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$5x + 6y = 0 \Rightarrow x = \left(-\frac{6}{5}\right) y$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{6}{5}y \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}$$

So an eigenvector for $\lambda = -4$ is $\begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix} \overset{\times 5}{\sim} \begin{bmatrix} -6 \\ 5 \end{bmatrix}$.

It is ok to multiply an eigenvector by any (nonzero) number

9. Diagonalization

Usually you see the above question worded differently:

Example 16:

Find D diagonal and P such that $A = PDP^{-1}$ where

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$\lambda = 7 \text{ or } \lambda = -4 \Rightarrow D = \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \qquad \text{Diagonal}$$

Note: Ok to write $D = \begin{bmatrix} -4 & 0 \\ 0 & 7 \end{bmatrix}$ as long as you put in the eigenvectors in the correct order.

For $\lambda = 7$ we found $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and for $\lambda = -4$ we found $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$ and so

$$P = \begin{bmatrix} 1 & -6 \\ 1 & 5 \end{bmatrix}$$

Here the first column of P *has* to be an eigenvector for $\lambda = 7$ (and the second column has to be an eigenvector for $\lambda = -4$), since we chose D in that order.

Interpretation: $A = PDP^{-1}$ means that A is "similar to" or "like" D

Example 17:

Find D diagonal and P such that $A = PDP^{-1}$ where

$$A = \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}$$

STEP 1: Eigenvalues

$$\begin{aligned}\left|A-\lambda I\right| &= \begin{vmatrix} 7-\lambda & -3 \\ 10 & -4-\lambda \end{vmatrix} \\ &= (7-\lambda)(-4-\lambda)-(-3)(10) \\ &= -28-7\lambda+4\lambda+\lambda^2+30 \\ &= \lambda^2-3\lambda+2 \\ &= (\lambda-1)(\lambda-2) \\ &= 0\end{aligned}$$

$$\lambda = 1 \text{ or } \lambda = 2 \Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{Diagonal}$$

STEP 2: $\lambda = 1$

$$\begin{aligned}\text{Nul } (A-1I) &= \left[\begin{array}{cc|c} 7-1 & -3 & 0 \\ 10 & -4-1 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 6 & -3 & 0 \\ 10 & -5 & 0 \end{array}\right] \\ &\xrightarrow{(\div 3)R_1\ (\div 5)R_2} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]\end{aligned}$$

Hence $2x - y = 0$ so $y = 2x$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 1$

STEP 3: $\lambda = 2$

$$\text{Nul }(A - 2I) = \left[\begin{array}{cc|c} 7-2 & -3 & 0 \\ 10 & -4-2 & 0 \end{array}\right]$$
$$= \left[\begin{array}{cc|c} 5 & -3 & 0 \\ 10 & -6 & 0 \end{array}\right]$$
$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 5 & -3 & 0 \\ 10-2(5) & -6-2(-3) & 0 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{cc|c} 5 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $5x - 3y = 0$

Faster way: (in 2 dimensions) $x = 3$ and $y = 5$ solve the equation, so an eigenvector for $\lambda = 2$ is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

STEP 4: Answer: $A = PDP^{-1}$ where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$