# THE WRONSKIAN

In this handout, we discuss a useful tool called the Wronskian, which helps us solve both linear ODE and systems of ODE

## 1. Motivation

**Example 1:**

Look at $y'' - 5y' + 6y = 0$

**Aux:** $r^2 - 5r + 6 = 0 \Rightarrow r = 2$ or $r = 3$

Strictly speaking, all this says is that $e^{2t}$ and $e^{3t}$ are solutions.

**Question:** How do we go from there to $y = Ae^{2t} + Be^{3t}$ ?

**Main Idea:** Start from $e^{2t}$ and $e^{3t}$ as building blocks, and "build up" our solution by using linear combinations

**Linear Fact:**

For linear homogeneous ODE:

(1) A constant times a solution is still a solution

(2) The sum of two solutions is still a solution.

Starting from $e^{2t}$ and $e^{3t}$, (1) says that $Ae^{2t}$ and $Be^{3t}$ are solutions for any $A$ and $B$, and (2) says that the sum $Ae^{2t}+Be^{3t}$ is a solution.

Could there be other solutions? *In theory* the answer could be yes, but it turns out that the answer is no: those are indeed all the solutions. The reason for this uses an important tool called the **Wronskian**:

## 2. The Wronskian

**Definition:**

The **Wronskian** of $f$ and $g$ is

$$W(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$$

**Note:** This is sometimes written as $W[f(t), g(t)]$

**Example 2:**

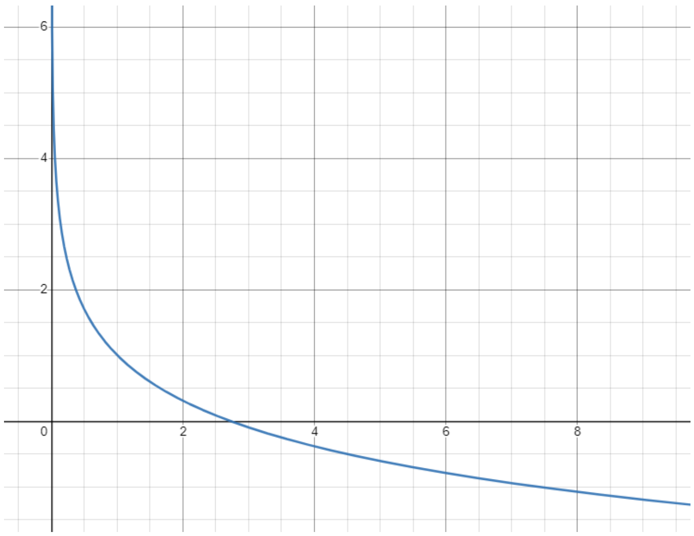
Find the Wronskian of $f(t) = t^2$ and $g(t) = t^3$

$$W(t) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = \left(t^2\right)\left(3t^2\right) - \left(t^3\right)(2t) = 3t^4 - 2t^4 = t^4$$

**Example 3:**

Same but $f(t) = t$ and $g(t) = \ln(t)$

$$W(t) = \begin{vmatrix} t & \ln(t) \\ 1 & \frac{1}{t} \end{vmatrix} = t\left(\frac{1}{t}\right) - \ln(t)(1) = 1 - \ln(t)$$




**Note:** For *arbitrary* functions, the Wronskian can change sign, like $W(t) = 1 - \ln(t)$ above. But **if** our functions solve an ODE, then something truly special happens:

Remember that for $y'' - 5y' + 6y = 0$ we had the solutions $e^{2t}$ and $e^{3t}$

Let's look at their Wronskian:

**Example 4:**

Find the Wronskian of $f(t) = e^{2t}$ and $g(t) = e^{3t}$

$$W(t) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{2t}\left(3e^{3t}\right) - \left(2e^{2t}\right)e^{3t} = 3e^{5t} - 2e^{5t} = e^{5t} \neq 0$$

Notice here that the Wronskian of $e^{2t}$ and $e^{3t}$ is *never* 0.

On the other hand, the general solution of the ODE is $y = Ae^{2t} + Be^{3t}$.

The miracle is that those two remarks are related:

**Wronskian Miracle:**

**If** $f(t)$ and $g(t)$ solve a second order ODE, then

(1) The Wronskian $W(t)$ is either *always* zero or *never* zero

(2) If $W(t)$ is never zero, the general solution to the ODE is

$$y = Af(t) + Bg(t)$$

Here: The Wronskian of $e^{2t}$ and $e^{3t}$ is $W(t) = e^{5t} \neq 0$, so (2) says that the general solution is $y = Ae^{2t} + Be^{3t}$. There are no other solutions.

Here is an example where the Wronskian is always zero.

**Example 5:**

Find the Wronskian of $f(t) = e^{2t}$ and $g(t) = 3e^{2t}$

$$W(t) = \begin{vmatrix} e^{2t} & 3e^{2t} \\ 2e^{2t} & 6e^{2t} \end{vmatrix} = e^{2t}\left(6e^{2t}\right) - \left(3e^{2t}\right)\left(2e^{2t}\right) = 6e^{4t} - 6e^{4t} = 0$$

Here one function is a multiple of the other. This is always true for linear ODE: If the Wronskian is 0, then one function is a multiple of the other one (more generally: *linearly dependent*), and if it is never zero, then they are *linearly independent*.

## 3. Fundamental Solutions

Because of the Wronskian Miracle above, it is useful to look for solutions of ODE whose Wronskian is nonzero. They have their own special name:

**Definition:**

We say $f(t)$ and $g(t)$ are **fundamental solutions** if:

(1) They solve a second-order linear ODE

(2) Their Wronskian $W(t)$ is nonzero

(Technically it's called a fundamental sol *set* since they come in pairs)

**Example 6:**

$e^{2t}$ and $e^{3t}$ are fundamental solutions of $y'' - 5y' + 6y = 0$

**Example 7:**

Show that $\cos(t)$ and $\sin(t)$ are fundamental solutions of $y''+y = 0$ and solve that ODE.

**STEP 1:** Check $\cos(t)$ and $\sin(t)$ solve $y'' + y = 0$

$$(\cos(t))'' + \cos(t) = (-\sin(t))' + \cos(t) = -\cos(t) + \cos(t) = 0\checkmark$$

Similarly for $\sin(t)$

**STEP 2:**

$$W(t) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1 \neq 0$$

**STEP 3:** From the Wronskian miracle above we have

$$y = A\cos(t) + B\sin(t)$$

**Example 8:**

Same but $\sqrt{t}$ and $\frac{1}{t}$ for $2t^2y'' + 3ty' - y = 0$ (with $t > 0$)

**STEP 1:** Check solutions. Notice $\sqrt{t} = t^{\frac{1}{2}}$

$$\begin{aligned}
&2t^2\left(t^{\frac{1}{2}}\right)'' + 3t\left(t^{\frac{1}{2}}\right)' - t^{\frac{1}{2}} \\
=&2t^2\left(\frac{1}{2}t^{-\frac{1}{2}}\right)' + 3t\frac{1}{2}t^{-\frac{1}{2}} - t^{\frac{1}{2}} \\
=&t^2\left(-\frac{1}{2}\right)t^{-\frac{3}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \\
=&-\frac{1}{2}t^{\frac{1}{2}} + \frac{3}{2}t^{\frac{1}{2}} - t^{\frac{1}{2}} \\
=&0
\end{aligned}$$

Similarly for $\frac{1}{t} = t^{-1}$

**STEP 2:** Wronskian

$$W(t) = \begin{vmatrix} \sqrt{t} & \frac{1}{t} \\ \frac{1}{2\sqrt{t}} & -\frac{1}{t^2} \end{vmatrix} = \sqrt{t}\left(-\frac{1}{t^2}\right) - \frac{1}{t}\left(\frac{1}{2\sqrt{t}}\right) = -\frac{1}{t\sqrt{t}} - \frac{1}{2t\sqrt{t}} = -\frac{3}{2t\sqrt{t}} \neq 0$$

Therefore $\sqrt{t}$ and $\frac{1}{t}$ are fundamental solutions

**STEP 3:** General Solution

$$y = A\sqrt{t} + B\left(\frac{1}{t}\right)$$

## 4. Abel's Formula

**Video:** Abel's Formula

Here is the coolest fact you need to know about the Wronskian:

How cool is it that the Wronskian, a tool used to solve ODE, *itself* solves a differential equation!

Suppose $y'' + Py' + Qy = 0$, then:

**Abel's Formula**

$$W' + PW = 0$$

**Mnemonic:** $W' +$ **P**ass**W**ord $= 0$

We can solve this using the integrating factor $e^{\int P}$ to get:

$$W = Ce^{-\int P}$$

(Beware of the minus sign!!)

**Consequence:** This formula explains why the Wronskian is either always zero (if $C = 0$) or never zero (if $C \neq 0$)

**Two Applications:** First, we can use this to find the Wronskian without knowing the solutions $f(t)$ and $g(t)$ beforehand.

**Example 9: (Application 1)**

Find $W(t)$ where $\cos(t)y'' + \sin(t)y' - ty = 0$

**STEP 1:** Divide by $\cos(t)$

$$y'' + \left(\frac{\sin(t)}{\cos(t)}\right) y' - \left(\frac{t}{\cos(t)}\right) y = 0$$

$P(t) = \frac{\sin(t)}{\cos(t)} = \tan(t)$

**STEP 2:** By Abel's Formula, we get

$$W(t) = Ce^{-\int P} = Ce^{-\int \tan(t)dt} = Ce^{-\ln\sec(t)} = \frac{C}{e^{\ln\sec(t)}} = \frac{C}{\sec(t)} = C\cos(t)$$

($C$ depends on which solutions we have)

More importantly, given *one* solution to a differential equation, we can use this to find *another* solution:

**Example 10: (Application 2)**

Suppose one solution to $t^2y'' + ty' - y = 0$ is $f(t) = t$. Find another solution $g(t)$ and then find the general solution $y$

**STEP 1:** Divide by $t^2$:

$$y'' + \left(\frac{1}{t}\right) y' - \left(\frac{1}{t^2}\right) y = 0$$

$$P(t) = \frac{1}{t}$$

**STEP 2:** By Abel's Formula:

$$W(t) = Ce^{-\int P} = Ce^{-\int \frac{1}{t}} = Ce^{-\ln(t)} = \frac{C}{e^{\ln(t)}} = \frac{C}{t}$$

**Note:** Since we're looking for *one* solution, we can let $C = 1$, and so

$$W(t) = \frac{1}{t}$$

**STEP 3:** Use the definition of $W(t)$ with $f(t) = t$ and $g(t)$ TBA

$$W(t) = \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} = tg'(t) - g(t) \overset{\text{WANT}}{=} \frac{1}{t}$$

This is a differential equation for $g(t)$, which we can now solve:

**STEP 4:** Solve this using integrating factors

$$\begin{aligned}
tg' - g =& \frac{1}{t} \\
g' + \left(-\frac{1}{t}\right) g =& \frac{1}{t^2}
\end{aligned}$$

$$P = -\frac{1}{t} \Rightarrow e^{\int P} = e^{\int -\frac{1}{t}} = e^{-\ln(t)} = \frac{1}{t}$$

$$\begin{aligned}
\left(\frac{1}{t}\right) g' - \left(\frac{1}{t}\right)\left(\frac{1}{t}\right) g =& \left(\frac{1}{t}\right)\left(\frac{1}{t^2}\right) \\
\left(\left(\frac{1}{t}\right) g\right)' =& \frac{1}{t^3} \\
\left(\frac{1}{t}\right) g =& \int t^{-3} = \frac{t^{-2}}{-2} = -\frac{1}{2t^2} \\
g(t) =& t\left(-\frac{1}{2t^2}\right) = -\frac{1}{2t}
\end{aligned}$$

**STEP 5:** From last time, we then get that the general solution is

$$y = Af(t) + Bg(t) = At + B\left(-\frac{1}{2t}\right) = At + \frac{B}{t}$$

(Since $B$ is just an arbitrary constant, so the $-\frac{1}{2}$ gets absorbed in $B$)

**Note:** You can get some pretty unexpected solutions from this!

**Example 11: (see Video below)**

One solution to $y'' - \tan(t)y' + 2y = 0$ is $f(t) = \sin(t)$. Using this method, another solution is

$$g(t) = -1 + \sin(t)\coth^{-1}(\sin(t))$$

**Video:** Abel's Formula

## 5. Systems

All the theory from second order equations translates almost verbatim to systems. In particular, here again the Wronskian appears

**Example 12:**

Find **fundamental solutions** and a **fundamental matrix** of $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$$

**STEP 1: Eigenvalues**

$$\begin{aligned}|A-\lambda I| = &\begin{vmatrix}1-\lambda & 1\\ 5 & -3-\lambda\end{vmatrix}\\ =&(1-\lambda)(-3-\lambda)-5\\ =&-3-\lambda+3\lambda+\lambda^2-5\\ =&\lambda^2+2\lambda-8\\ =&(\lambda+4)(\lambda-2)=0\end{aligned}$$

Which gives $\lambda = -4$ and $\lambda = 2$

**STEP 2:** $\lambda = -4$

$$\text{Nul }(A-(-4)I) = \left[\begin{array}{cc|c}1-(-4) & 1 & 0\\ 5 & -3-(-4) & 0\end{array}\right] = \left[\begin{array}{cc|c}5 & 1 & 0\\ 5 & 1 & 0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}5 & 1 & 0\\ 0 & 0 & 0\end{array}\right]$$

$5x + y = 0 \Rightarrow y = -5x$

$$\mathbf{x} = \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\\ -5x\end{bmatrix} = x\begin{bmatrix}1\\ -5\end{bmatrix}$$

**STEP 3:** $\lambda = 2$

$$\begin{aligned}\text{Nul }(A-2I) = &\left[\begin{array}{cc|c}1-2 & 1 & 0\\ 5 & -3-2 & 0\end{array}\right]\\ =&\left[\begin{array}{cc|c}-1 & 1 & 0\\ 5 & -5 & 0\end{array}\right]\\ \to&\left[\begin{array}{cc|c}-1 & 1 & 0\\ 1 & -1 & 0\end{array}\right]\\ \to&\left[\begin{array}{cc|c}-1 & 1 & 0\\ 0 & 0 & 0\end{array}\right]\end{aligned}$$

$-x + y = 0 \Rightarrow y = x$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**STEP 4: Solution**

$$\mathbf{x}(t) = C_1 e^{-4t} \begin{bmatrix} 1 \\ -5 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} e^{-4t} \\ -5e^{-4t} \end{bmatrix} + C_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

**Definition: (Fundamental Solutions)**

$$\mathbf{x_1}(t) = \begin{bmatrix} e^{-4t} \\ -5e^{-4t} \end{bmatrix} \text{ and } \mathbf{x_2}(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

**Means:**

(1) $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ solve the ODE

(2) $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are linearly independent (not multiples of each other)

The fundamental matrix is just putting this into a matrix:

**Definition: (Fundamental Matrix)**

$$\Psi(t) = \begin{bmatrix} e^{-4t} & e^{2t} \\ -5e^{-4t} & e^{2t} \end{bmatrix}$$

**Means:** The columns of $\Psi(t)$ are fundamental solutions, they satisfy (1) and (2).

The determinant of $\Psi(t)$ is called the **Wronskian**

**Definition: (Wronskian)**

$$W(t) = \begin{vmatrix} e^{-4t} & e^{2t} \\ -5e^{-4t} & e^{2t} \end{vmatrix} = e^{-4t}e^{2t} - e^{2t}(-5e^{-4t}) = e^{-2t} + 5e^{-2t} = 6e^{-2t}$$

**Fact:**

For solutions of linear ODE, $W(t)$ is either *always* 0 or *never* 0.

In the second case, the general solution of the ODE is

$$C_1\mathbf{x_1}(t) + C_2\mathbf{x_2}(t)$$

Where $\mathbf{x_1}$ and $\mathbf{x_2}$ are the columns of $\Psi(t)$

Here $W(t) \neq 0$ and so the general solution is

$$\mathbf{x}(t) = C_1 \begin{bmatrix} e^{-4t} \\ -5e^{-4t} \end{bmatrix} + C_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = C_1 e^{-4t} \begin{bmatrix} 1 \\ -5 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**Example 13:**

Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} 2/t & -1/t \\ 3/t & -2/t \end{bmatrix} \mathbf{x}$$

Assume that $\mathbf{x_1} = \begin{bmatrix} t \\ t \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 1/t \\ 3/t \end{bmatrix}$ solve the ODE

$$W(t) = \begin{vmatrix} t & 1/t \\ t & 3/t \end{vmatrix} = t\left(\frac{3}{t}\right) - \left(\frac{1}{t}\right)t = 3 - 1 = 2 \neq 0$$

The Wronskian is nonzero, and therefore the general solution is

$$\mathbf{x}(t) = C_1\mathbf{x_1}(t) + C_2\mathbf{x_2}(t) = C_1\begin{bmatrix} t \\ t \end{bmatrix} + C_2\begin{bmatrix} 1/t \\ 3/t \end{bmatrix}$$

## 6. Abel's Formula

The amazing thing is that, once again, the Wronskian $W(t)$ satisfies its own differential equation:

**Definition: (Trace)**

$$\text{Tr}\begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix} = 2 + 8 = 10 \qquad \text{(Sum of Diagonal Terms)}$$

**Abel's Formula**

For solutions of ODE, the Wronskian satisfies

$$W'(t) = \text{Tr}(A)W(t)$$

**Example 14:**

$$\mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{bmatrix} 1 & 1 \\ 5 & -3 \end{bmatrix}$$

$$W'(t) = (1-3)W(t) \Rightarrow W'(t) = -2W(t) \Rightarrow W(t) = Ce^{-2t}$$

And in fact we previously found that $W(t) = 6e^{-2t}$

This again shows that the Wronskian is either always zero or never zero.

**Example 15:**

$$\text{Find } W(t) \text{ where } \mathbf{x}' = \begin{bmatrix} 3/t & -2/t \\ 2/t & -2/t \end{bmatrix} \mathbf{x}$$

$$W'(t) = \left(\frac{3}{t} - \frac{2}{t}\right) W(t) = \frac{1}{t} W(t)$$

$$W'(t) - \frac{1}{t} W(t) = 0$$

**Integrating Factor:** $e^{-\int \frac{1}{t} dt} = e^{-\ln(t)} = \frac{1}{e^{\ln(t)}} = \frac{1}{t}$

$$\begin{aligned} \left(\frac{1}{t}\right) W'(t) + \left(\frac{1}{t}\right)\left(-\frac{1}{t}\right) W(t) &= 0 \\ \left(\frac{1}{t} W(t)\right)' &= 0 \\ \frac{1}{t} W(t) &= C \\ W(t) &= Ct \end{aligned}$$

**Application:** Given one solution $\mathbf{x_1}$, we can use this to find another solution $\mathbf{x_2}$:

**Example 16:**

One solution of the ODE

$$\mathbf{x}' = \begin{bmatrix} 3/t & -2/t \\ 2/t & -2/t \end{bmatrix} \mathbf{x}$$

Is $\mathbf{x_1} = \begin{bmatrix} 1/t \\ 2/t \end{bmatrix}$ Find another solution $\mathbf{x_2}$ and solve the ODE

Assume the other solution is $\mathbf{x_2} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$

**STEP 1:** From before, we know that $W(t) = Ct$, and so $W(t) = t$ (just need one solution).

$$\text{But } W(t) = \begin{vmatrix} 1/t & f(t) \\ 2/t & g(t) \end{vmatrix} = t$$

$$\begin{aligned} \left(\frac{1}{t}\right) g(t) - f(t)\left(\frac{2}{t}\right) =& t \\ g(t) - 2f(t) =& t^2 \\ g(t) =& t^2 + 2f(t) \end{aligned}$$

$$\mathbf{x_2} = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \begin{bmatrix} f(t) \\ t^2 + 2f(t) \end{bmatrix}$$

**STEP 2:** Plug this into the ODE

$$\mathbf{x_2}' = \begin{bmatrix} 3/t & -2/t \\ 2/t & -2/t \end{bmatrix} \mathbf{x_2}$$

$$\begin{bmatrix} f'(t) \\ 2t + 2f'(t) \end{bmatrix} = \begin{bmatrix} 3/t & -2/t \\ 2/t & -2/t \end{bmatrix} \begin{bmatrix} f(t) \\ t^2 + 2f(t) \end{bmatrix}$$

The first line gives us

$$\begin{aligned} f'(t) &= \left(\frac{3}{t}\right) f(t) - \left(\frac{2}{t}\right)\left(t^2 + 2f(t)\right) \\ f'(t) &= \left(\frac{3}{t}\right) f(t) - 2t - \left(\frac{4}{t}\right) f(t) \\ f'(t) &= -\frac{1}{t} f(t) - 2t \\ f'(t) + &\left(\frac{1}{t}\right) f(t) = -2t \end{aligned}$$

**STEP 3:** Integrating factor

$$e^{\int \frac{1}{t} dt} = e^{\ln(t)} = t$$

$$\begin{aligned} t f'(t) + t \left(\frac{1}{t}\right) f(t) =& t\,(-2t) \\ (t f(t))' =& -2t^2 \\ t f(t) =& -\frac{2}{3} t^3 \\ f(t) =& -\frac{2}{3} t^2 \end{aligned}$$

$$g(t) = t^2 + 2f(t) = t^2 + 2\left(-\frac{2}{3}t^2\right) = t^2 - \frac{4}{3}t^2 = -\frac{1}{3}t^2$$

**STEP 4:** $\mathbf{x_2}(t)$

$$\mathbf{x_2}(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix} = \begin{bmatrix} (-2/3)\,t^2 \\ (-1/3)\,t^2 \end{bmatrix} \overset{\times(-3)}{\sim} \begin{bmatrix} 2t^2 \\ t^2 \end{bmatrix} = t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

**STEP 5: General Solution**

$$\mathbf{x}(t) = C_1 \mathbf{x_1}(t) + C_2 \mathbf{x_2}(t) = C_1 \left(\frac{1}{t}\right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 t^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$