LECTURE: EXISTENCE AND UNIQUENESS

Let's continue our exploration of existence and uniqueness:

1. Another Crazy Example

Video: Dangerous Differential Equation

Example 1:

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

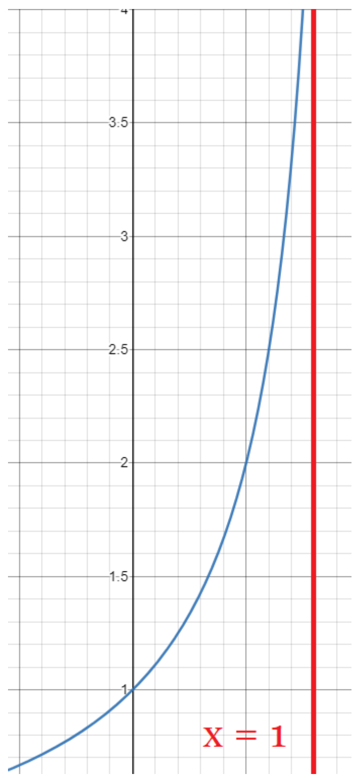
$$\begin{aligned} y' &= y^2 \\ \frac{y'}{y^2} &= 1 \\ \left(-\frac{1}{y}\right)' &= 1 \\ -\frac{1}{y} &= t + C \end{aligned}$$

$$y = -\frac{1}{t+C}$$

$$y(0) = 1 \Rightarrow -\frac{1}{0+C} = 1 \Rightarrow -\frac{1}{C} = 1 \Rightarrow C = -1$$

$$y = -\left(\frac{1}{t-1}\right) = \frac{1}{1-t}$$

Here the solution blows up at $t = 1$



While solutions can exist for short time (near $t = 0$), they can still blow up for *large* time, at $t = 1$, like bacteria that are short-lived.

2. Existence and Uniqueness

Isn't this depressing? What *is* true about ODE then? When *do* we have a unique solution? This is the point of the celebrated ODE Existence/Uniqueness Theorem:

Existence/Uniqueness Theorem:

Consider the ODE

$$\begin{cases} y' = f(y,t) \\ y(0) = y_0 \end{cases}$$

Where y_0 is a given number

If f and $\frac{\partial f}{\partial y}$ are continuous, then the ODE has a unique solution $y = y(t)$ for t close enough to 0

In other words, as long as f is smooth enough, there *is* a unique solution, at least for short time

Remarks:

(1) $\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y
(you differentiate f with respect to y, treating t as a constant)

(2) "Close enough to 0" means there is some $\delta > 0$ such that $y(t)$ exists on $(-\delta, \delta)$

(3) It's actually enough for f and $\frac{\partial f}{\partial y}$ to be continuous near $(0, y_0)$

(4) There is nothing special about 0, this theorem holds for any initial time t_0

Example 2:

Does the following ODE has a unique solution?

$$\begin{cases} y' = (y+t)^2 \sin(t) \\ y(10) = -5 \end{cases}$$

$f(y,t) = (y+t)^2 \sin(t)$, which is continuous, and furthermore

$$\frac{\partial f}{\partial y} = 2(y+t)\sin(t) \text{ is continuous}$$

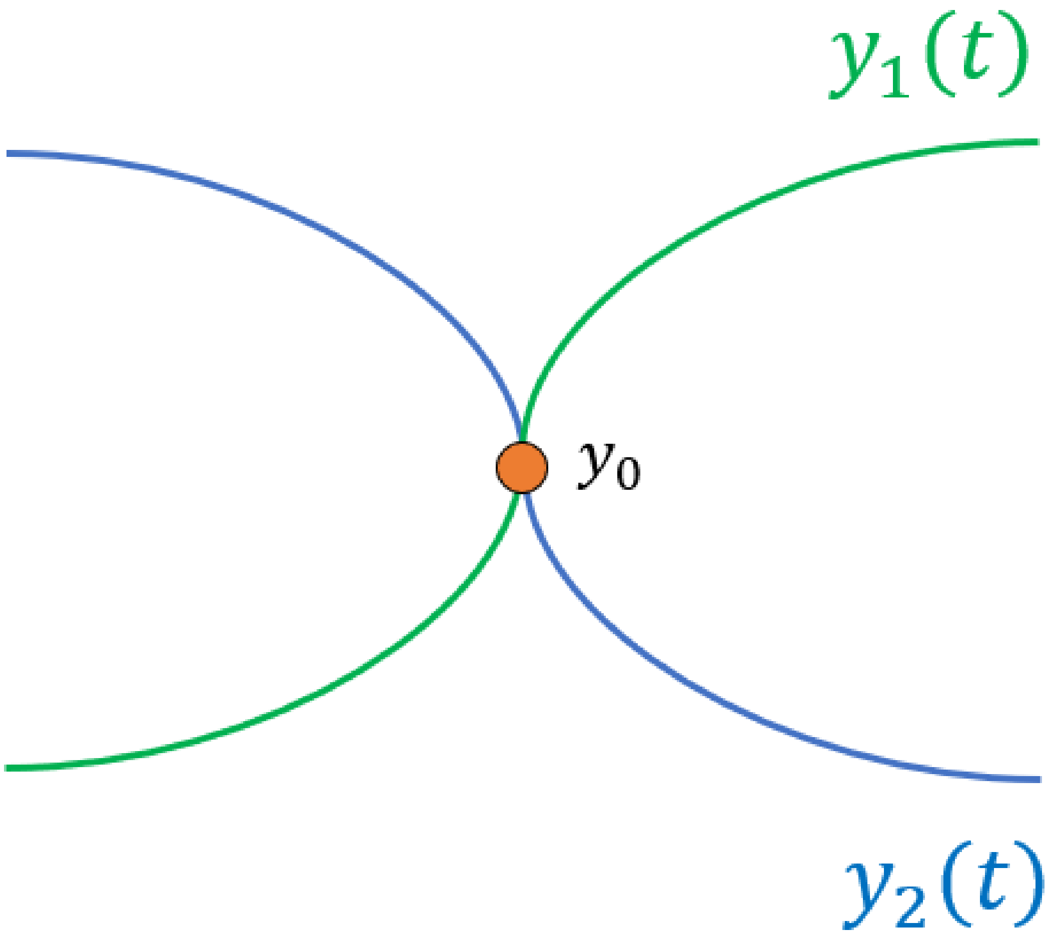
So the answer is **YES** at least for some short time around $t = 10$

What went wrong in the examples that we discussed?

For $y' = \sqrt{y}$ the theorem fails because even though $f(y,t) = \sqrt{y}$ continuous, $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$ is not continuous near $y = 0$.

For $y' = y^2$ the theorem is technically true, but remember it guarantees only existence for *small* t. We don't know anything about existence for large t (there*are* theorems though that guarantee long-time existence)

Cool Application: Because of uniqueness, solutions of $y' = f(y,t)$ can never cross! Because if they did, we can take the point of intersection as the initial value y_0 and then get two solutions with the same initial condition, like in the picture below:



A couple of words about the proof:

STEP 1: Transform the differential equation into an *integral equation:*

$$\begin{aligned} y' &= f(y,t) \\ \int_0^t y'(s)ds &= \int_0^t f(y(s),s)ds \\ y(t) - y(0) &= \int_0^t f(y,s)ds \\ y(t) &= y_0 + \int_0^t f(y,s)ds \end{aligned}$$

In general, differential equations and integral equations are equivalent

STEP 2: Define the right-hand-side as

$$g(y) = y_0 + \int_0^t f(y,s)ds$$

And show that $g(y) = y$, that is g has a "fixed point."

STEP 3: To do this, you use the "Banach Fixed Point Theorem," which is a theorem (outside the scope of this class) that tells us when a function has a fixed point. It's in that theorem where we use the continuity of f and $\frac{\partial f}{\partial y}$.

Intuitively, you need continuity of $\frac{\partial f}{\partial y}$ to show that f doesn't spread out points too much, we want to make sure that if $|y_1 - y_2|$ is small, then so is $|f(y_1, s) - f(y_2, s)|$

Check out the video below for details and if you want to learn more about the proof.

Video: ODE Existence/Uniqueness Theorem

Note: Fixed point theorems are really cool, here are a couple of fun applications:

(1) If you shake a snowglobe, there is one snowflake that didn't change position

(2) There is one point on a world map that points exactly where your finger points

(3) (related) There are two opposite points on earth that have the same pressure and temperature (called the Borsuk-Ulam Theorem)

3. Separation of Variables: Motivation

Let's finally solve some differential equations!

Goal: Solve separable equations, which are equations of the form

$$\frac{dy}{dt} = \frac{f(t)}{g(y)}$$

Example 3:

$$\frac{dy}{dt} = \frac{2t}{y^2}$$

Note: Below we'll find a faster way of doing this

STEP 1: Put all the y on one side and all the t on the other side:

$$y^2 \left(\frac{dy}{dt}\right) = 2t$$

STEP 2: Integrate each side with respect to t

$$\int y^2 \left(\frac{dy}{dt}\right) dt = \int 2t dt$$

For the left-hand-side, use the $u-$sub $y = y(t)$ then $dy = \left(\frac{dy}{dt}\right) dt$ and this becomes

$$\int y^2 dy = \int 2t dt$$

(Notice the power of the chain rule/u-sub here)

Now you can find anti-derivatives and get

$$\begin{aligned}\left(\frac{1}{3}\right) y^3 + C =& t^2 + C' \\ \left(\frac{1}{3}\right) y^3 =& t^2 + \underbrace{C' - C}_{C} \\ y^3 =& 3t^2 + \underbrace{3C}_{C} \\ y =& \sqrt[3]{3t^2 + C}\end{aligned}$$

4. Separation of Variables

Here is a faster way of doing this:

Example 4:

$$\begin{cases} t^2 \left(\frac{dy}{dt}\right) =& (1 - t^2)(y^2 + 1) \\ y(1) =& 0 \end{cases}$$

STEP 1: cross-multiply, treating dy and dt as separate entities:

$$\begin{aligned} t^2 dy &= \left(1-t^2\right)\left(y^2+1\right) dt \\ \frac{dy}{y^2+1} &= \left(\frac{1-t^2}{t^2}\right) dt \\ \int \left(\frac{1}{y^2+1}\right) dy &= \int \frac{1}{t^2} - 1 dt \\ \tan^{-1}(y) &= -\frac{1}{t} - t + C \\ y &= \tan\left(-\frac{1}{t} - t + C\right) \end{aligned}$$

STEP 2: To solve for C, use $y(1) = 0$, so $t = 1$ and $y = 0$

$\tan(-1-1+C) = 0 \Rightarrow \tan(-2+C) = 0 \Rightarrow -2+C = \pi m \Rightarrow C = 2+\pi m$

$$y = \tan\left(-\frac{1}{t} - t + 2 + \pi m\right) = \tan\left(-\frac{1}{t} - t + 2\right)$$

Remark: The two methods *seem* different, but they're actually the same. In fact, if you used the first method, you would have obtained

$$\int \frac{1}{y^2+1}\left(\frac{dy}{dt}\right) dy = \int \frac{1}{t^2} - 1 dt \Rightarrow \int \frac{1}{y^2+1} dy = \int \frac{1}{t^2} - 1 dt$$

Which is the same result that we got here.