LECTURE: MIDTERM 2 – REVIEW

1. Laplace Transform

Example 1:

$$\begin{cases} y' - 3y = 6f(t) \\ y(0) = 2 \end{cases} \quad \text{where } f(t) = \begin{cases} 0 \text{ if } 0 \leq t < 4 \\ e^{t-4} \text{ if } t \geq 4 \end{cases}$$

STEP 1: Take Laplace transforms

Note: $f(t) = e^{t-4}u_4(t)$

$$\begin{aligned} \mathcal{L}\{y'\} - 3\mathcal{L}\{y\} &= 6\mathcal{L}\left\{e^{t-4}u_4(t)\right\} \\ s\mathcal{L}\{y\} - y(0) - 3\mathcal{L}\{y\} &= 6e^{-4s}\mathcal{L}\left\{e^t\right\} \qquad \text{use } \mathcal{L}\{f(t-c)u_c\} = e^{-cs}\mathcal{L}\{f\} \\ (s-3)\mathcal{L}\{y\} - 2 &= \frac{6e^{-4s}}{s-1} \\ (s-3)\mathcal{L}\{y\} &= 2 + \frac{6e^{-4s}}{s-1} \\ \mathcal{L}\{y\} &= \frac{2}{s-3} + \frac{6e^{-4s}}{(s-1)(s-3)} \end{aligned}$$

STEP 2: Partial Fractions

$$\frac{6}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3} = \frac{A(s-3) + B(s-1)}{(s-1)(s-3)} = \frac{(A+B)s - 3A - B}{(s-1)(s-3)}$$

$$\begin{cases} A + B = 0 \\ -3A - B = 6 \end{cases} \Rightarrow \begin{cases} B = -A \\ -3A + A = 6 \end{cases} \Rightarrow \begin{cases} B = -A \\ -2A = 6 \end{cases} \Rightarrow \begin{cases} A = -3 \\ B = 3 \end{cases}$$

STEP 3:

$$\begin{aligned}\mathcal{L}\left\{y\right\} =& \frac{2}{s-3} + \left(\frac{-3}{s-1} + \frac{3}{s-3}\right)e^{-4s}\\ =& \mathcal{L}\left\{2e^{3t}\right\} + \mathcal{L}\left\{-3e^{t} + 3e^{3t}\right\}e^{-4s}\\ =& \mathcal{L}\left\{2e^{3t} + \left(-3e^{(t-4)} + 3e^{3(t-4)}\right)u_4(t)\right\}\end{aligned}$$

$$y = 2e^{3t} + \left(-3e^{(t-4)} + 3e^{3(t-4)}\right)u_4(t)$$

2. Inverse Laplace Transform

Example 2:

Find a function whose Laplace transform is

$$\frac{e^{-2s}}{4s^2+12s+25}$$

STEP 1: Cannot factor out the denominator, so complete the square:

$$\begin{aligned}4s^2+12s+25 =& 4\left(s^2+3s+\frac{25}{4}\right) = 4\left[\left(s+\frac{3}{2}\right)^2 - \frac{9}{4} + \frac{25}{4}\right]\\ =& 4\left[\left(s+\frac{3}{2}\right)^2 + \frac{16}{4}\right] = 4\left[\left(s+\frac{3}{2}\right)^2 + 4\right]\end{aligned}$$

$$\frac{e^{-2s}}{4s^2+12s+25} = \frac{1}{4}e^{-2s}\left(\frac{1}{\left(s+\frac{3}{2}\right)^2+4}\right)$$

STEP 2: The term in parentheses is a shifted version by $-\frac{3}{2}$ units of

$$\frac{1}{s^2+4} = \frac{1}{2}\left(\frac{2}{s^2+4}\right) = \frac{1}{2}\mathcal{L}\left\{\sin(2t)\right\} = \mathcal{L}\left\{\frac{1}{2}\sin(2t)\right\}$$

$$\text{Hence } \frac{1}{\left(s+\frac{3}{2}\right)^2+4} = \mathcal{L}\left\{e^{-\frac{3}{2}t}\left(\frac{1}{2}\right)\sin(2t)\right\}$$

STEP 3:

$$\begin{aligned}\frac{1}{4}e^{-2s}\left(\frac{1}{\left(s+\frac{3}{2}\right)^2+4}\right) &= \frac{1}{4}e^{-2s}\mathcal{L}\left\{e^{-\frac{3}{2}t}\left(\frac{1}{2}\right)\sin(2t)\right\}\\ &= e^{-2s}\mathcal{L}\left\{\frac{1}{8}e^{-\frac{3}{2}t}\sin(2t)\right\}\\ &= \mathcal{L}\left\{\frac{1}{8}e^{-\frac{3}{2}(t-2)}\sin(2(t-2))u_2(t)\right\}\end{aligned}$$

$$\textbf{Answer: } f(t) = \frac{1}{8}e^{-\frac{3}{2}(t-2)}\sin(2(t-2))u_2(t)$$

3. Var of Par

Example 3:

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

STEP 1: Homogeneous Solution

$$\textbf{Aux: } r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$$

$$y_0(t) = Ae^t + Bte^t$$

STEP 2: Var of Par

$$y_p(t) = u(t)e^t + v(t)te^t$$

$$\begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^t}{1+t^2} \end{bmatrix}$$

Note: Here we used $(te^t)' = e^t + te^t = (1+t)e^t$

Denominator:

$$\begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^t(t+1)e^t - te^te^t = (t+1)e^{2t} - te^{2t} = e^{2t}$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{1+t^2} & (t+1)e^t \end{vmatrix}}{e^{2t}} = \frac{0 - (te^t)\left(\frac{e^t}{1+t^2}\right)}{e^{2t}} = \frac{\frac{-te^{2t}}{1+t^2}}{e^{2t}} = \frac{-t}{1+t^2}$$

$$v'(t) = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{1+t^2} \end{vmatrix}}{e^{2t}} = \frac{(e^t)\left(\frac{e^t}{1+t^2}\right) - 0}{e^{2t}} = \frac{\frac{e^{2t}}{1+t^2}}{e^{2t}} = \frac{1}{1+t^2}$$

$$u(t) = \int \frac{-t}{1+t^2} dt = -\frac{1}{2}\ln\left(1+t^2\right)$$

(Here we used the $u-$sub $u = 1+t^2$)

$$v(t) = \int \frac{1}{1+t^2} dt = \tan^{-1}(t)$$

$$y_p(t) = -\frac{1}{2}\ln\left(1+t^2\right)e^t + \tan^{-1}(t)te^t$$

$$y(t) = y_0(t) + y_p(t) = Ae^t + Bte^t - \frac{1}{2}\ln\left(1+t^2\right)e^t + \tan^{-1}(t)te^t$$

4. Who's that particular solution?

Example 4:

Guess the form of the particular solution

(a) $y'' - 4y' + 13y = e^{2t}$

$$\begin{aligned} r^2 - 4r + 13 &= 0 \\ (r-2)^2 - 4 + 13 &= 0 \\ (r-2)^2 + 9 &= 0 \\ r - 2 &= \pm 3i \\ r &= 2 \pm 3i \end{aligned}$$

$$y_0 = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$$

e^{2t} corresponds to $r = 2$, which does not coincide, so

$$y_p = Ae^{2t}$$

(b) $y'' - 4y' + 13y = \cos(3t)$

$\cos(3t)$ corresponds to $r = 3i$, which does not coincide, so

$$y_p = A\cos(3t) + B\sin(3t)$$

(c) $y'' - 4y' + 13y = e^{2t}\sin(3t)$

$e^{2t}\sin(3t)$ corresponds to $r = 2 + 3i$, which coincides, so

$$y_p = Ate^{2t}\cos(3t) + Bte^{2t}\sin(3t)$$

(d) $y'' - 4y' + 13y = t^2\cos(3t)$

$\cos(3t)$ corresponds to $r = 3i$ which does not coincide, so

$$y_p = (At^2 + Bt + C)\cos(3t) + (Dt^2 + Et + F)\sin(3t)$$

(e) $y'' - 4y' + 13y = te^{2t}\sin(3t)$

$e^{2t}\sin(3t)$ corresponds to $r = 2 + 3i$ which coincides, so

$$y_p = t(At + B)e^{2t}\cos(3t) + t(Ct + D)e^{2t}\sin(3t)$$

5. More Laplace Transforms

Example 5: (more practice)

$$\begin{cases} y'' - 7y' + 12y = \delta(t-2) \\ y(0) = 0 \\ y'(0) = -2 \end{cases}$$

STEP 1: Take Laplace Transforms of the ODE

$$\begin{aligned}
\mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 12\mathcal{L}\{y\} =& \mathcal{L}\{\delta(t-2)\} \\
\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) - 7\left(s\mathcal{L}\{y\} - y(0)\right) + 12\mathcal{L}\{y\} =& e^{-2s} \\
s^2\mathcal{L}\{y\} - (-2) - 7s\mathcal{L}\{y\} + 12\mathcal{L}\{y\} =& e^{-2s} \\
\left(s^2 - 7s + 12\right)\mathcal{L}\{y\} =& -2 + e^{-2s} \\
\mathcal{L}\{y\} = \left(\frac{1}{s^2 - 7s + 12}\right)\left(-2 + e^{-2s}\right)
\end{aligned}$$

STEP 2: Partial Fractions

Note: $s^2 - 7s + 12 = (s-3)(s-4)$

$$\begin{aligned}
\frac{1}{s^2 - 7s + 12} =& \frac{A}{s-3} + \frac{B}{s-4} \\
=& \frac{A(s-4) + B(s-3)}{(s-3)(s-4)} \\
=& \frac{As - 4A + Bs - 3B}{s^2 - 7s + 12} \\
=& \frac{(A+B)s + (-4A - 3B)}{s^2 - 7s + 12}
\end{aligned}$$

$$\begin{cases} A + B = 0 \\ -4A - 3B = 1 \end{cases}$$

The first equation gives $B = -A$ and the second equation becomes

$$-4A - 3(-A) = 1 \Rightarrow -4A + 3A = 1 \Rightarrow -A = 1 \Rightarrow A = -1$$

And $B = -A = 1$

$$\frac{1}{s^2 - 7s + 12} = \frac{-1}{s-3} + \frac{1}{s-4}$$

STEP 3:

$$\begin{aligned}\mathcal{L}\left\{y\right\} &= \left(\frac{-1}{s-3} + \frac{1}{s-4}\right)\left(-2 + e^{-2s}\right) \\ &= \mathcal{L}\left\{-e^{3t} + e^{4t}\right\}\left(-2 + e^{-3s}\right) \\ &= -2\mathcal{L}\left\{-e^{3t} + e^{4t}\right\} + e^{-2s}\mathcal{L}\left\{-e^{3t} + e^{4t}\right\} \\ &= \mathcal{L}\left\{2e^{3t} - 2e^{4t} + \left(-e^{3(t-2)} + e^{4(t-2)}\right) u_2(t)\right\}\end{aligned}$$

STEP 4: Answer:

$$y = 2e^{3t} - 2e^{4t} + \left(-e^{3(t-2)} + e^{4(t-2)}\right) u_2(t)$$

6. Undetermined Coefficients

Example 6: (more practice)

$$y'' - 6y' - 7y = (-32t - 20)e^{-t}$$

STEP 1: Homogeneous Solution

Aux: $r^2 - 6r - 7 = 0 \Rightarrow (r-7)(r+1) = 0 \Rightarrow r = -1$ or $r = 7$

$$y_0(t) = Ae^{-t} + Be^{7t}$$

STEP 2: Undetermined Coefficients:

Since $e^{-t} \rightsquigarrow r = -1$ which is one of the roots, we have resonance.

Moreover, $(2t - 1) \rightsquigarrow (At + B)$ and so we guess

$$y_p(t) = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}$$

$$
\begin{aligned}
(y_p)'(t) &= (2At+B)e^{-t} + (At^2+Bt)\left(-e^{-t}\right) \\
&= \left(2At + B - At^2 - Bt\right)e^{-t} \\
&= \left(-At^2 + (2A-B)t + B\right)e^{-t}
\end{aligned}
$$

$$
\begin{aligned}
(y_p)''(t) &= \left(-2At + (2A-B)\right)e^{-t} + \left(-At^2 + (2A-B)t + B\right)\left(-e^{-t}\right) \\
&= \left(-2At + 2A - B + At^2 - 2At + Bt - B\right)e^{-t} \\
&= \left(At^2 - 4At + Bt + 2A - 2B\right)e^{-t}
\end{aligned}
$$

$$
\begin{aligned}
(y_p)'' - 6(y_p)' - 7(y_p) &= (-32t-20)e^{-t} \\
\left(At^2 - 4At + Bt + 2A - 2B\right)\cancel{e^{-t}} - 6\left(-At^2 + (2A-B)t + B\right)\cancel{e^{-t}} & \\
-7\left(At^2 + Bt\right)\cancel{e^{-t}} &= (-32t-20)\cancel{e^{-t}} \\
\cancel{At^2} - 4At + \cancel{Bt} + 2A - 2B + \cancel{6At^2} - 12At + \cancel{6Bt} - 6B \cancel{- 7At^2} - \cancel{7Bt} &= -32t - 20 \\
-16At + (2A - 8B) &= -32t - 20
\end{aligned}
$$

$$
\begin{cases} -16A = -32 \\ 2A - 8B = -20 \end{cases} \Rightarrow \begin{cases} A = 2 \\ -8B = -20 - 2A = -24 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 3 \end{cases}
$$

$$
y_p(t) = (2t^2 + 3t)e^{-t}
$$

STEP 3: General Solution

$$
y = y_0 + y_p = Ae^{-t} + Be^{7t} + (2t^2 + 3t)e^{-t}
$$