APMA 0350 – MIDTERM 1 – REVIEW PROBLEMS

1. Integrating Factors

Example 1:

$$t^2 y' + t(t+2)y = e^t$$

Note: Generally, use integrating factors if you can put your equation in the form $y' + Py =$ Junk

STEP 1: Standard Form

$$\frac{t^2 y'}{t^2} + \left(\frac{t(t+2)}{t^2}\right) y = \frac{e^t}{t^2}$$
$$y' + \left(\frac{t+2}{t}\right) y = \frac{e^t}{t^2}$$

STEP 2: Integrating Factor

$$P = \frac{t+2}{t}$$

$$e^{\int P} = e^{\int \frac{t+2}{t}} = e^{\int 1 + \frac{2}{t}} = e^{t + 2\ln(t)} = e^t \left(e^{\ln(t)}\right)^2 = t^2 e^t$$

STEP 3: Multiply

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$$
\begin{aligned}
t^2e^t\left(y' + \left(\frac{t+2}{t}\right)y\right) =& t^2e^t\frac{e^t}{t^2}\\
\left(t^2e^ty\right)' =& e^{2t}\\
t^2e^ty =& \int e^{2t}\\
t^2e^ty =& \frac{1}{2}e^{2t} + C\\
y =& \frac{1}{t^2e^t}\left(\frac{1}{2}e^{2t} + C\right)\\
y =& \frac{e^t}{2t^2} + \frac{C}{t^2}e^{-t}
\end{aligned}
$$

Example 2:

$$
\begin{cases}
t^2y' + 2ty = \ln(t)\\
y(1) = 2
\end{cases}
$$

STEP 1: Standard Form

$$
y' + \left(\frac{2t}{t^2}\right)y = \frac{\ln(t)}{t^2} \Rightarrow y' + \left(\frac{2}{t}\right)y = \frac{\ln(t)}{t^2}
$$

STEP 2: Integrating Factor

$$
e^{\int P} = e^{\int \frac{2}{t}dt} = e^{2\ln(t)} = \left(e^{\ln(t)}\right)^2 = t^2
$$

$$\begin{aligned} t^2\left(y' + \frac{2}{t}y\right) =& t^2\left(\frac{\ln(t)}{t^2}\right) \\ \left(t^2 y\right)' =& \ln(t) \\ t^2 y =& \int \ln(t)dt = t\ln(t) - t + C \\ y =& \frac{t\ln(t) - t + C}{t^2} \end{aligned}$$

STEP 3: Initial Condition

$$y(1) = 2 \Rightarrow \frac{1\ln(1) - 1 + C}{1^2} = 2 \Rightarrow -1 + C = 2 \Rightarrow C = 3$$

$$y = \frac{t\ln(t) - t + 3}{t^2}$$

2. Separation of Variables

Example 3:

$$\begin{cases} \sqrt{x^2+1}\left(\frac{dy}{dx}\right) =& 4x(y^2+1) \\ y(0) =& 1 \end{cases}$$

Use Separation of variables when you can put all the y on one side and all the x on the other side.

STEP 1: Cross multiply

$$\begin{aligned}\sqrt{x^2+1}\,dy =& 4x(y^2+1)dx \\ \frac{dy}{y^2+1} =& \frac{4x}{\sqrt{x^2+1}}dx \\ \int \frac{dy}{y^2+1} =& \int \frac{4x}{\sqrt{x^2+1}}dx\end{aligned}$$

STEP 2: Integrate

$$\int \frac{dy}{y^2+1}dy = \tan^{-1}(y)$$

$$\int \frac{4x}{\sqrt{x^2+1}}dx = \int \frac{2}{\sqrt{u}}du = \int 2u^{-\frac{1}{2}}du = 2\left(2u^{\frac{1}{2}}\right) = 4\sqrt{x^2+1}+C$$

$$(u = x^2+1, du = 2xdx \Rightarrow 4xdx \Rightarrow 2du)$$

STEP 3:

$$\tan^{-1}(y) = 4\sqrt{x^2+1}+C$$

$$y = \tan\left(4\sqrt{x^2+1}+C\right)$$

STEP 4: Initial Condition

$$\begin{aligned} y(0) =& 1 \\ \tan\left(4\sqrt{0+1}+C\right) =& 1 \\ \tan(4+C) =& 1 \\ 4+C =& \tan^{-1}(1) \\ 4+C =& \frac{\pi}{4} \\ C =& \frac{\pi}{4}-4 \end{aligned}$$

$$y = \tan\left(4\sqrt{x^2+1} + \frac{\pi}{4} - 4\right)$$

3. Exact Equations

Example 4:

$$\left(2\cos(y) + 2y^2\right) + \left(-x\sin(y) + 2xy\right) y' = 0$$

Hint: Multiply by x first

STEP 1:

$$x\left(2\cos(y) + 2y^2\right) + x\left(-x\sin(y) + 2xy\right)\frac{dy}{dx} = x(0)$$
$$\left(2x\cos(y) + 2xy^2\right)dx + \left(-x^2\sin(y) + 2x^2y\right)dy = 0$$

STEP 2: Check conservative

$$P_y = \left(2x\cos(y) + 2xy^2\right)_y = -2x\sin(y) + 4xy$$
$$Q_x = \left(-x^2\sin(y) + 2x^2y\right)_x = -2x\sin(y) + 4xy \checkmark$$

STEP 3: Find f

$$f(x,y) = \int 2x\cos(y) + 2xy^2 dx = x^2\cos(y) + x^2y^2 + \text{ Junk}$$
$$f(x,y) = \int -x^2\sin(y) + 2x^2 y dy = -x^2\left(-\cos(y)\right) + 2x^2\left(\frac{y^2}{2}\right)$$
$$= x^2\cos(y) + x^2y^2 + \text{ Junk}$$

$$f(x,y) = x^2\cos(y) + x^2y^2$$

STEP 4: Solution

$$x^2\cos(y) + x^2y^2 = C \Rightarrow x^2\left(\cos(y) + y^2\right) = C$$

Example 5:

$$\begin{cases} \frac{dy}{dx} = \frac{-ye^{xy}}{2y + xe^{xy}} \\ y(0) = 1 \end{cases}$$

Leave your answer in implicit form

STEP 1: Cross multiply

$$(2y + xe^{xy})\, dy = -\, ye^{xy} dx$$
$$ye^{xy} dx + (2y + xe^{xy})\, dy = 0$$

STEP 2: Check exact

$$P = ye^{xy}$$
$$Q = 2y + xe^{xy}$$

$$P_y = \left(ye^{xy}\right)_y = e^{xy} + ye^{xy}(x) = e^{xy} + xye^{xy}$$

$$Q_x = \left(2y + xe^{xy}\right)_x = e^{xy} + xe^{xy}(y) = e^{xy} + xye^{xy}$$

STEP 3: Integrate

$$f_x = P \Rightarrow f = \int P dx = \int ye^{xy} dx = \int e^u du = e^u + \text{ Junk } = e^{xy} + \text{ Junk}$$

$$u = xy \Rightarrow du = y dx$$

$$f_y = Q \Rightarrow f = \int Q dy = \int 2y + xe^{xy} dy = y^2 + e^{xy} + \text{ Junk}$$

$$f(x, y) = y^2 + e^{xy}$$

$$y^2 + e^{xy} = C$$

STEP 4: Initial Condition

$y(0) = 1 \Rightarrow x = 0$ and $y = 1$

$$1^2 + e^{(0)(1)} = C \Rightarrow 1 + 1 = C \Rightarrow C = 2$$

$$y^2 + e^{xy} = 2$$

4. Second Order ODE

Example 6:

$$\begin{cases} y'' - 6y' + 25y = 0 \\ y(0) = 2 \\ y'(0) = -2 \end{cases}$$

STEP 1: Auxiliary Equation

$$\begin{aligned} r^2 - 6r + 25 &= 0 \\ r &= \frac{6 \pm \sqrt{(-6)^2 - 4(1)(25)}}{2} \\ &= \frac{6 \pm \sqrt{36 - 100}}{2} \\ &= \frac{6 \pm \sqrt{-64}}{2} \\ &= \frac{6 \pm 8i}{2} \\ &= 3 \pm 4i \end{aligned}$$

STEP 2: General Solution

$$y = Ae^{3t}\cos(4t) + Be^{3t}\sin(4t)$$

STEP 3: Initial Condition

$$\begin{aligned} y(0) &= 2 \\ Ae^{0}\cos(0) + Be^{0}\sin(0) &= 2 \\ A &= 2 \end{aligned}$$

$$\begin{aligned} y &= 2e^{3t}\cos(4t) + Be^{3t}\sin(4t) \\ y' &= 2\left(3e^{3t}\right)\cos(4t) + 2e^{3t}\left(-4\sin(4t)\right) + B\left(3e^{3t}\right)\sin(4t) + Be^{3t}\left(4\cos(4t)\right) \end{aligned}$$

$$\begin{aligned} y'(0) &= -2 \\ 6e^{0}\cos(0) - 8e^{0}\sin(0) + 3Be^{0}\sin(0) + 4Be^{0}\cos(0) &= -2 \\ 6 + 4B &= -2 \\ 4B &= -8 \\ B &= -2 \end{aligned}$$

$$y = 2e^{3t}\cos(4t) - 2e^{3t}\sin(4t)$$

Example 7:

$$y'' - 4y' - 21y = 0$$

With the additional assumption that $\lim_{t\to\infty} y = 0$

Aux: $r^2 - 4r - 21 = (r+3)(r-7) = 0 \Rightarrow r = -3$ or $r = 7$

$$y = Ae^{-3t} + Be^{7t} \to A(0) + B(\infty) = B(\infty)$$

In order to have a limit of 0, we must have $B = 0$ and so

$$y = Ae^{-3t}$$

And indeed, y goes to 0 as $t \to \infty$ ✓

Example 8:

Solve the ODE whose auxiliary equation is

$$r^4(r-1)^2(r^2+1)(r^2+4r+5)^2 = 0$$

$r = 0$ (quadruple root), $r = 1$ (double), $r = \pm i$, $r = -2 \pm i$ (double)

$$\begin{aligned} y =& A + Bt + Ct^2 + Dt^3 \\ &+ Ee^t + Fte^t \\ &+ G\cos(t) + H\sin(t) \\ &+ Ie^{-2t}\cos(t) + Je^{-2t}\sin(t) + Kte^{-2t}\cos(t) + Lte^{-2t}\sin(t) \end{aligned}$$