APMA 0350 – EXAM 1 – SOLUTIONS

1. **Integrating Factors**

STEP 1: Standard Form

$$y' + \left(\frac{2}{t}\right) y = \frac{\cos(t)}{t^2}$$

STEP 2: Integrating Factor

$$e^{\int \frac{2}{t} dt} = e^{2\ln(t)} = \left(e^{\ln(t)}\right)^2 = t^2$$

Here we used $t > 0$ so $|t| = t$

STEP 3: Multiply by t^2

$$\begin{aligned} t^2 y' + t^2 \left(\frac{2}{t}\right) y =& t^2 \left(\frac{\cos(t)}{t^2}\right) \\ \left(t^2 y\right)' =& \cos(t) \\ t^2 y =& \int \cos(t) dt = \sin(t) + C \\ y =& \frac{\sin(t) + C}{t^2} \end{aligned}$$

STEP 4: Initial Condition

$$\begin{aligned} y(\pi) =& 1 \\ 1 =& \frac{\sin(\pi) + C}{\pi^2} \\ \frac{C}{\pi^2} =& 1 \\ C =& \pi^2 \end{aligned}$$

STEP 5: Solution

$$y = \frac{\sin(t) + \pi^2}{t^2}$$

2. Separation of variables

STEP 1:

$$\begin{aligned}\frac{dy}{dt} =& t\left(y^2\right)\left(1+t^2\right)^{-\frac{1}{2}}\\ \frac{dy}{y^2} =& \frac{t}{\sqrt{1+t^2}}dt\\ \int\frac{dy}{y^2} =& \int\frac{t}{\sqrt{1+t^2}}dt\\ -\frac{1}{y} =& \sqrt{1+t^2}+C \qquad (u=1+t^2)\\ y =& -\frac{1}{\sqrt{1+t^2}+C}\end{aligned}$$

STEP 2:

$$\begin{aligned}y(0) =& \frac{1}{3}\\ -\frac{1}{1+C} =& \frac{1}{3}\\ 1+C =& -3\\ C =& -4\end{aligned}$$

STEP 3: Solution

$$y = -\left(\frac{1}{\sqrt{1+t^2}-4}\right)$$

3. Exact Equations

STEP 1: Check Exact

$$
\begin{aligned}
P_y &= \left(ye^{2xy} + x\right)_y = e^{2xy} + ye^{2xy}(2x) = e^{2xy}\left(1 + 2xy\right) \\
Q_x &= \left(kxe^{2xy}\right)_x = ke^{2xy} + kxe^{2xy}(2y) = ke^{2xy}\left(1 + 2xy\right)
\end{aligned}
$$

The equation is exact iff $P_y = Q_x$ and so $\boxed{k = 1}$

STEP 2:

$$
\begin{aligned}
f_x =& ye^{2xy} + x \Rightarrow f = \int ye^{(2y)x} + x dx = \frac{ye^{2xy}}{2y} + \frac{x^2}{2} + g(y) = \frac{1}{2}e^{2xy} + \frac{x^2}{2} + g(y) \\
f_y =& xe^{2xy} \Rightarrow f = \int xe^{(2x)y} dy = \frac{xe^{2xy}}{2x} + h(x) = \frac{1}{2}e^{2xy} + h(x)
\end{aligned}
$$

Therefore $f(x, y) = \frac{1}{2}e^{2xy} + \frac{x^2}{2}$

STEP 3: Answer:

$$
\begin{aligned}
\frac{1}{2}e^{2xy} + \frac{x^2}{2} =& C \\
e^{2xy} + x^2 =& \underbrace{2C}_{C} \\
e^{2xy} =& C - x^2 \\
2xy =& \ln(C - x^2) \\
y =& \frac{1}{2x}\ln(C - x^2)
\end{aligned}
$$

4. Separation of Variables[1]

STEP 1:

$$
\begin{aligned}
\frac{dy}{dt} &= (\sin(t))\, y \\
\frac{dy}{y} &= \sin(t) dt \\
\int \frac{dy}{y} &= \int \sin(t) dt \\
\ln|y| &= -\cos(t) + C \\
|y| &= e^{-\cos(t) + C} \\
y &= \underbrace{\pm e^{C}}_{C} e^{-\cos(t)} \\
y &= C e^{-\cos(t)}
\end{aligned}
$$

STEP 2:

$$
\begin{aligned}
y(0) &= y_0 \\
C e^{-\cos(0)} &= y_0 \\
C e^{-1} &= y_0 \\
C &= e y_0
\end{aligned}
$$

$$
y = C e^{-\cos(t)} = e y_0 e^{-\cos(t)} = y_0 e^{1-\cos(t)}
$$

STEP 3:

$$
\begin{aligned}
y(0) &= 3y_0 \\
y_0 e^{1-\cos(t)} &= 3y_0 \\
e^{1-\cos(t)} &= 3 \\
1 - \cos(t) &= \ln(3) \\
\cos(t) &= 1 - \ln(3) \\
t &= \cos^{-1}(1 - \ln(3))
\end{aligned}
$$

[1] You could also use Integrating Factors here

5. **STEP 1: Prep Work**

$h = \frac{5-1}{2} = 2$

$t_0 = 1$ and $t_1 = 3$ and $t_2 = 5$

STEP 2:

$$
\begin{aligned}
y_0 =& y(1) = 3 \\
y_1 =& y_0 + hf(y_0, t_0) = 3 + 2\left(2y_0 - 1\right) = 3 + 2\left(2(3) - 1\right) = 3 + 10 = 13 \\
y_2 =& y_1 + hf(y_1, t_1) = 13 + 2\left(2(13) - 1\right) = 13 + 2(25) = 13 + 50 = 63
\end{aligned}
$$

Therefore an approximate value of $y(5)$ is 63

6. **STEP 1: Auxiliary Equation**

$$r^2-2r+5=0 \Rightarrow (r-1)^2-1+5=0 \Rightarrow (r-1)^2=-4 \Rightarrow r-1=\pm 2i \Rightarrow r=1\pm 2i$$

$$y = Ae^t\cos(2t) + Be^t\sin(2t)$$

STEP 2: Initial Condition

$$y\left(\frac{\pi}{2}\right) = Ae^{\frac{\pi}{2}}\cos(\pi) + Be^{\frac{\pi}{2}}\sin(\pi) = -Ae^{\frac{\pi}{2}} = 0 \Rightarrow A = 0$$

$$y = Be^t\sin(2t)$$

$$y' = Be^t\sin(2t) + 2Be^t\cos(2t)$$

$$y'\left(\frac{\pi}{2}\right) = Be^{\frac{\pi}{2}}\sin(\pi) + 2Be^{\frac{\pi}{2}}\cos(\pi) = -2Be^{\frac{\pi}{2}} = -2 \Rightarrow B = e^{-\frac{\pi}{2}}$$

STEP 3: Answer

$$y = e^{-\frac{\pi}{2}}e^t\sin(2t) = e^{t-\frac{\pi}{2}}\sin(2t)$$