APMA 0350 – MIDTERM 2 – SOLUTIONS

1. **Homogeneous Solution**

$$\begin{aligned} r^2 - 4r + 5 =& 0 \\ (r-2)^2 + 1 =& 0 \\ (r-2)^2 =& -1 \\ r - 2 =& \pm i \\ r =& 2 \pm i \end{aligned}$$

$$y_0 = Ae^{2t}\cos(t) + Be^{2t}\sin(t)$$

Particular Solution

(a) $\left(t^3 + 3t^2\right)e^{2t}$ corresponds to $r = 2$ so no resonance

$$y_p = \left(At^3 + Bt^2 + Ct + D\right)e^{2t}$$

(b) $(2t+3)\cos(t)$ corresponds to $r = \pm i$ so no resonance

$$y_p = (At + B)\cos(t) + (Ct + D)\sin(t)$$

(c) $te^{2t}\sin(t)$ corresponds to $r = 2 \pm i$ so resonance

$$y_p = t\,(At + B)\,e^{2t}\cos(t) + t\,(Ct + D)\,e^{2t}\sin(t)$$

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2. **STEP 1: Standard Form**

$$y'' - 6y' + 9y = \frac{e^{3t}}{t^2}$$

STEP 2: Homogeneous Solution

$$\textbf{Aux } r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r = 3$$

$$y_0 = Ae^{3t} + Bte^{3t}$$

STEP 2: Var of Par

$$y_p(t) = u(t)e^{3t} + v(t)te^{3t}$$

$$\begin{bmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (1+3t)e^{3t} \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{3t}}{t^2} \end{bmatrix}$$

Note: Here we used $\left(te^{3t}\right)' = e^{3t} + t\left(3e^{3t}\right) = (1+3t)e^{3t}$

Denominator:

$$\begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (1+3t)e^{3t} \end{vmatrix} = e^{3t}(3t+1)e^{3t} - te^{3t}\left(3e^{3t}\right) = (3t+1)e^{6t} - 3te^{6t} = e^{6t}$$

Using Cramer's rule, we get

$$u'(t) = \frac{\begin{vmatrix} 0 & te^{3t} \\ \frac{e^{3t}}{t^2} & (1+3t)e^{3t} \end{vmatrix}}{e^{6t}} = \frac{0 - \left(te^{3t}\right)\left(\frac{e^{3t}}{t^2}\right)}{e^{6t}} = -\frac{1}{t}$$

$$v'(t) = \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & \frac{e^{3t}}{t^2} \end{vmatrix}}{e^{6t}} = \frac{\left(e^{3t}\right)\left(\frac{e^{3t}}{t^2}\right) - 0}{e^{6t}} = \frac{1}{t^2}$$

$$u(t) = \int -\frac{1}{t}dt = -\ln(t)$$

$$v(t) = \int \frac{1}{t^2}dt = -\frac{1}{t}$$

$$y_p(t) = -\ln(t)e^{3t} + \left(-\frac{1}{t}\right)te^{3t} = -\ln(t)e^{3t} - e^{3t}$$

3. **STEP 1: Laplace Transforms**

$$\begin{aligned}
\mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} =& 6\,\mathcal{L}\{\delta(t-5)\} \\
\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) - 7\left(s\mathcal{L}\{y\} - y(0)\right) + 10\mathcal{L}\{y\} =& 6e^{-5s} \\
s^2\mathcal{L}\{y\} - 3s - 7s\mathcal{L}\{y\} + 21 + 10\mathcal{L}\{y\} =& 6e^{-5s} \\
\left(s^2 - 7s + 10\right)\mathcal{L}\{y\} - 3s + 21 =& 6e^{-5s} \\
\left(s^2 - 7s + 10\right)\mathcal{L}\{y\} =& 3s - 21 + 6e^{-5s} \\
\mathcal{L}\{y\} = \frac{3s-21}{(s-2)(s-5)} + \frac{6}{(s-2)(s-5)}e^{-5s}
\end{aligned}$$

STEP 2: Partial Fractions

$$\frac{3s-21}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5} = \frac{A(s-5)+B(s-2)}{(s-2)(s-5)} = \frac{(A+B)s + (-5A-2B)}{(s-2)(s-5)}$$

$$\begin{cases} A + B =3 \\ -5A - 2B = -21 \end{cases} \Rightarrow \begin{cases} B = 3 - A \\ -5A - 2(3-A) = -21 \end{cases} \Rightarrow \begin{cases} B = 3 - A \\ -3A - 6 = -21 \end{cases}$$

$$\begin{cases} B = 3 - A \\ A = \dfrac{-15}{-3} = 5 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = 3 - 5 = -2 \end{cases}$$

$$\frac{3s-21}{(s-2)(s-5)} = \frac{5}{s-2} - \frac{2}{s-5}$$

$$\frac{6}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5} = \frac{A(s-5)+B(s-2)}{(s-2)(s-5)} = \frac{(A+B)s + (-5A-2B)}{(s-2)(s-5)}$$

$$\begin{cases} A + B = 0 \\ -5A - 2B = 6 \end{cases} \Rightarrow \begin{cases} B = -A \\ -5A + 2A = 6 \end{cases} \Rightarrow \begin{cases} B = -A \\ -3A = 6 \end{cases}$$

$$\begin{cases} A = -2 \\ B = 2 \end{cases}$$

$$\frac{6}{(s-2)(s-5)} = -\frac{2}{s-2} + \frac{2}{s-5}$$

STEP 3:

$$
\begin{aligned}
\mathcal{L}\left\{y\right\} =& \frac{5}{s-2} - \frac{2}{s-5}\left(-\frac{2}{s-2} + \frac{2}{s-5}\right)e^{-5s} \\
=& \mathcal{L}\left\{5e^{2t} - 2e^{5t}\right\} + \mathcal{L}\left\{-2e^{2t} + 2e^{5t}\right\}e^{-5s} \\
=& \mathcal{L}\left\{5e^{2t} - 2e^{5t} + \left(-2e^{2(t-5)} + 2e^{5(t-5)}\right)u_5(t)\right\}
\end{aligned}
$$

Answer:

$$
y = 5e^{2t} - 2e^{5t} + \left(-2e^{2(t-5)} + 2e^{5(t-5)}\right)u_5(t)
$$

4. The equation is of the form

$$\phi - (t \star \phi) = 1 + t$$

Take Laplace transforms

$$\begin{aligned}
\mathcal{L}\{\phi\} - \mathcal{L}\{t \star \phi\} &= \mathcal{L}\{1+t\} \\
\mathcal{L}\{\phi\} - \mathcal{L}\{t\}\mathcal{L}\{\phi\} &= \frac{1}{s} + \frac{1}{s^2} \\
\mathcal{L}\{\phi\} - \left(\frac{1}{s^2}\right)\mathcal{L}\{\phi\} &= \frac{s+1}{s^2} \\
\mathcal{L}\{\phi\}\left(1 - \frac{1}{s^2}\right) &= \frac{s+1}{s^2} \\
\mathcal{L}\{\phi\}\left(\frac{s^2-1}{s^2}\right) &= \frac{s+1}{s^2} \\
\mathcal{L}\{\phi\} &= \frac{s+1}{s^2-1} = \frac{s+1}{(s+1)(s-1)} \\
\mathcal{L}\{\phi\} &= \frac{1}{s-1} = \mathcal{L}\left\{e^t\right\}
\end{aligned}$$

$$\phi = e^t$$