APMA 0350 – EXAM 3 – SOLUTIONS

1. **STEP 1: Eigenvalues**

$$\begin{aligned}\left|A-\lambda I\right| &= \begin{vmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{vmatrix} \\ &= (6-\lambda)(4-\lambda)-(-1)(5) \\ &= 24-6\lambda-4\lambda+\lambda^2+5 \\ &= \lambda^2-10\lambda+29 \\ &= (\lambda-5)^2-25+29 \\ &= (\lambda-5)^2+4\end{aligned}$$

$$(\lambda-5)^2=-4 \Rightarrow \lambda-5=\pm 2i \Rightarrow \lambda = 5\pm 2i$$

STEP 2: $\lambda = 5+2i$

$$\begin{aligned}\text{Nul }(A-(5+2i)I) &= \left[\begin{array}{cc|c} 6-(5+2i) & -1 & 0 \\ 5 & 4-(5+2i) & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right] \\ &\longrightarrow \left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]\end{aligned}$$

$$(1-2i)x-y=0 \Rightarrow y=(1-2i)x$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (1-2i)x \end{bmatrix} = x\begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$$

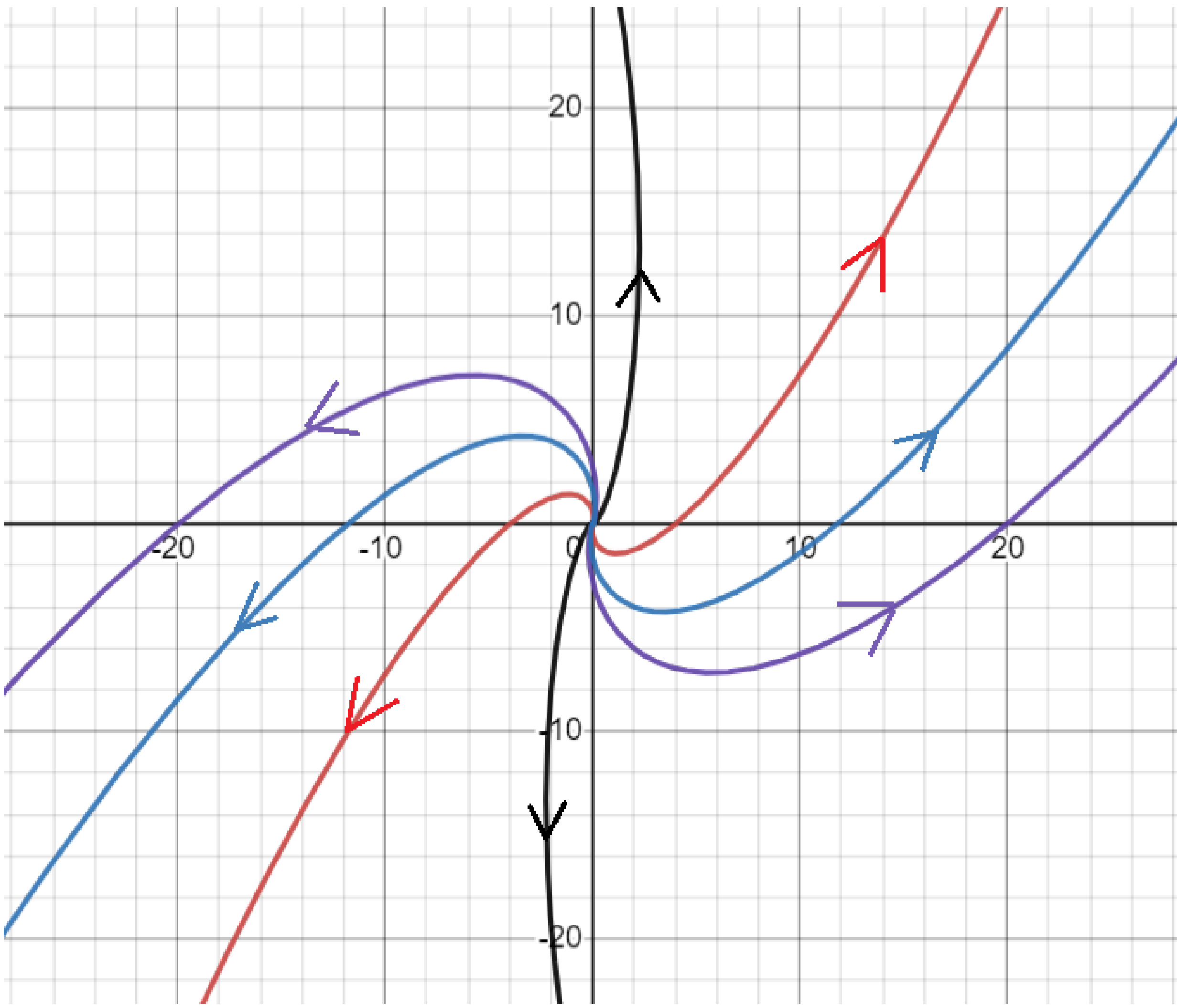
STEP 3: Solution

$$\begin{aligned}e^{(5+2i)t}\begin{bmatrix} 1 \\ 1-2i \end{bmatrix} &= \left(e^{5t}e^{2ti}\right)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}+i\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) \\ &= e^{5t}\left(\cos(2t)+i\sin(2t)\right)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}+i\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) \\ &= e^{5t}\left(\cos(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}-\sin(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix}\right) \\ &+ie^{5t}\left(\cos(2t)\begin{bmatrix} 0 \\ -2 \end{bmatrix}+\sin(2t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\end{aligned}$$

$$\begin{aligned}\mathbf{x}(t) =& C_1 e^{5t}\left(\cos(2t)\begin{bmatrix}1\\1\end{bmatrix} - \sin(2t)\begin{bmatrix}0\\-2\end{bmatrix}\right)\\ &+C_2 e^{5t}\left(\cos(2t)\begin{bmatrix}0\\-2\end{bmatrix} + \sin(2t)\begin{bmatrix}1\\1\end{bmatrix}\right)\end{aligned}$$

STEP 4: Phase Portrait

Because of the e^{5t} term, solutions are spiraling outwards



2. **STEP 1: Eigenvalues**

$$\begin{aligned}\left|A-\lambda I\right| = &\begin{vmatrix}-6-\lambda & 4\\ -1 & -2-\lambda\end{vmatrix}\\ =&(-6-\lambda)(-2-\lambda)-(4)(-1)\\ =&12+6\lambda+2\lambda+\lambda^2+4\\ =&\lambda^2+8\lambda+16\\ =&(\lambda+4)^2=0\end{aligned}$$

Which gives $\lambda = -4$ (repeated)

STEP 2:

$$\begin{aligned}e^{At} =&e^{-4t}e^{(A+4I)t}\\ =&e^{-4t}\left(I+(A+4I)t\right)\\ =&e^{-4t}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}-6+4 & 4\\ -1 & -2+4\end{bmatrix}t\right)\\ =&e^{-4t}\begin{bmatrix}1-2t & 4t\\ -t & 1+2t\end{bmatrix}\end{aligned}$$

STEP 3:

$$\mathbf{x}(t) = C_1e^{-4t}\begin{bmatrix}1-2t\\ -t\end{bmatrix}+C_2e^{-4t}\begin{bmatrix}4t\\ 1+2t\end{bmatrix}$$

STEP 4: Initial Condition: $\mathbf{x}(0) = \begin{bmatrix}2\\-3\end{bmatrix}$

Here use $\mathbf{x}(t) = e^{At}\mathbf{x}(0) = e^{At}\begin{bmatrix}2\\-3\end{bmatrix}$

$$\mathbf{x}(t) = 2e^{-4t}\begin{bmatrix}1-2t\\ -t\end{bmatrix}-3e^{-4t}\begin{bmatrix}4t\\ 1+2t\end{bmatrix} = e^{-4t}\begin{bmatrix}2-4t-12t\\ -2t-3-6t\end{bmatrix} = e^{-4t}\begin{bmatrix}2-16t\\ -3-8t\end{bmatrix}$$

3. **Eigenvalues:**

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix}$$
$$= (1-\lambda)(3-\lambda) - (2)(4)$$
$$= 3 - \lambda - 3\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 4\lambda - 5$$
$$= (\lambda - 5)(\lambda + 1) = 0$$

This gives $\lambda = 5$ or $\lambda = -1$

(a)

$$\mathbf{f} = \begin{bmatrix} e^{2t} \\ 3e^{2t} \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightsquigarrow \lambda = 2$$

This doesn't coincide, so there is no resonance and we guess

$$\mathbf{x_p} = e^{2t} \begin{bmatrix} A \\ B \end{bmatrix}$$

(b)

$$\mathbf{f} = \begin{bmatrix} 7te^{3t} \\ 4e^{-4t} \end{bmatrix} = e^{3t} \begin{bmatrix} 7t \\ 0 \end{bmatrix} + e^{-4t} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \rightsquigarrow \lambda = 3, -4$$

This doesn't coincide, so there is no resonance and we guess

$$\mathbf{x_p} = e^{3t} \left(\begin{bmatrix} A \\ B \end{bmatrix} t + \begin{bmatrix} C \\ D \end{bmatrix} \right) + e^{-4t} \begin{bmatrix} E \\ F \end{bmatrix}$$

(c)

$$\mathbf{f} = \begin{bmatrix} 2e^{5t} \\ e^{5t} \end{bmatrix} = e^{5t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightsquigarrow \lambda = 5$$

This coincides, so there is resonance and we guess

$$\mathbf{x_p} = e^{5t} \left(\begin{bmatrix} A \\ B \end{bmatrix} t + \begin{bmatrix} C \\ D \end{bmatrix} \right)$$

(d)

$$\mathbf{f} = \begin{bmatrix} e^{5t}\cos(t) \\ e^{3t}\sin(2t) \end{bmatrix} = e^{5t}\cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t}\sin(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \lambda = 5 \pm i, 3 \pm 2i$$

This doesn't coincide, so there is no resonance and we guess

$$\mathbf{x_p} = e^{5t}\cos(t)\begin{bmatrix} A \\ B \end{bmatrix} + e^{5t}\sin(t)\begin{bmatrix} C \\ D \end{bmatrix} + e^{3t}\cos(2t)\begin{bmatrix} E \\ F \end{bmatrix} + e^{3t}\sin(2t)\begin{bmatrix} G \\ H \end{bmatrix}$$

(e)

$$\mathbf{f} = \begin{bmatrix} 2e^{5t} \\ e^{-t} \end{bmatrix} = 2e^{5t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \lambda = 5, -1$$

This coincides, so there is resonance, and we guess

$$\mathbf{f} = \left(\begin{bmatrix} A \\ B \end{bmatrix} t + \begin{bmatrix} C \\ D \end{bmatrix}\right) e^{5t} + \left(\begin{bmatrix} E \\ F \end{bmatrix} t + \begin{bmatrix} G \\ H \end{bmatrix}\right) e^{-t}$$

4. **STEP 1:** Eigenvalues

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} \\ &= (2-\lambda)(-2-\lambda) - (-5)(1) \\ &= -4 - 2\lambda + 2\lambda + \lambda^2 + 5 \\ &= \lambda^2 + 1\end{aligned}$$

Which gives $\lambda = \pm i$

STEP 2: $\lambda = i$

$$\text{Nul } (A - iI) = \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 1 & -2-i & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2-i & -5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $(2-i)x - 5y = 0 \Rightarrow x = 5$ and $y = 2 - i$ works

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2-i \end{bmatrix}$$

STEP 4: Homogeneous Solution

$$\begin{aligned}e^{it}\begin{bmatrix} 5 \\ 2-i \end{bmatrix} &= (\cos(t) + i\sin(t))\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) \\ &= \left(\cos(t)\begin{bmatrix} 5 \\ 2 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) + i\left(\cos(t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin(t)\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + i\begin{bmatrix} 5\sin(t) \\ -\cos(t) + 2\sin(t) \end{bmatrix}\end{aligned}$$

$$\mathbf{x_0}(t) = C_1\begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + C_2\begin{bmatrix} 5\sin(t) \\ -\cos(t) + 2\sin(t) \end{bmatrix}$$

STEP 5: Variation of Parameters

$$\mathbf{x_p}(t) = u(t)\begin{bmatrix} 5\cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + v(t)\begin{bmatrix} 5\sin(t) \\ -\cos(t) + 2\sin(t) \end{bmatrix}$$

$$\begin{bmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t) + \sin(t) & -\cos(t) + 2\sin(t) \end{bmatrix}\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 5\sec(t) \\ 0 \end{bmatrix}$$

Denominator:

$$\begin{aligned}
&\begin{vmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & -\cos(t)+2\sin(t) \end{vmatrix} \\
=&5\cos(t)\left(-\cos(t)+2\sin(t)\right) - 5\sin(t)\left(2\cos(t)+\sin(t)\right) \\
=&-5\cos^2(t)+10\cos(t)\sin(t)-10\cos(t)\sin(t)-5\sin^2(t) \\
=&-5\left(\cos^2(t)+\sin^2(t)\right) \\
=&-5
\end{aligned}$$

$$\begin{aligned}
u'(t) =& \frac{\begin{vmatrix} 5\sec(t) & 5\sin(t) \\ 0 & -\cos(t)+2\sin(t) \end{vmatrix}}{-5} \\
=& \left(-\frac{1}{5}\right)\left(-\cos(t)+2\sin(t)\right) 5\sec(t) \\
=& \cos(t)\sec(t) - 2\sin(t)\sec(t) \\
=&1-2\tan(t)
\end{aligned}$$

$$u(t) = \int 1 - 2\tan(t) = t - 2\ln|\sec(t)|$$

$$\begin{aligned}
v'(t) =& \frac{\begin{vmatrix} 5\cos(t) & 5\sec(t) \\ 2\cos(t)+\sin(t) & 0 \end{vmatrix}}{-5} \\
=& \left(-\frac{1}{5}\right)\left(2\cos(t)+\sin(t)\right)\left(-5\sec(t)\right) \\
=&2\cos(t)\sec(t)+\sin(t)\sec(t) \\
=&2\cos(t)\left(\frac{1}{\cos(t)}\right)+\sin(t)\left(\frac{1}{\cos(t)}\right) \\
=&2+\tan(t)
\end{aligned}$$

$$v(t) = \int 2 + \tan(t) = 2t + \ln|\sec(t)|$$

STEP 6: Answer

$$\begin{aligned}
\mathbf{x_p}(t) =& \left(t - 2\ln|\sec(t)|\right)\begin{bmatrix} 5\cos(t) \\ 2\cos(t)+\sin(t) \end{bmatrix} \\
&+ \left(2t + \ln|\sec(t)|\right)\begin{bmatrix} 5\sin(t) \\ -\cos(t)+2\sin(t) \end{bmatrix}
\end{aligned}$$

5. **STEP 1:** Equilibrium Solutions

$$\begin{cases} x' = x(-1+2y) = 0 \\ y' = y - x^2 - y^2 = 0 \end{cases}$$

The first equation gives $x = 0$ or $-1 + 2y = 0 \Rightarrow y = \frac{1}{2}$

Case 1: $x = 0$

Then the second equation becomes

$$y - y^2 = 0 \Rightarrow y(1-y) = 0 \Rightarrow y = 0 \text{ or } y = 1$$

This gives the points $(0,0)$ and $(0,1)$

Case 2: $y = \frac{1}{2}$

Then the second equation becomes

$$\frac{1}{2} - x^2 - \frac{1}{4} = 0 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

This gives the points $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Equilibrium Points: $(0,0), (0,1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$

STEP 2: Classification

$$\nabla F(x,y) = \begin{bmatrix} -1+2y & 2x \\ -2x & 1-2y \end{bmatrix}$$

Case 1: $(0,0)$

$$\nabla F(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\lambda = -1 < 0$ and $\lambda = 1 > 0$, so $(0,0)$ is a **saddle**

Case 2: $(0,1)$

$$\nabla F(0,1) = \begin{bmatrix} -1+2 & 0 \\ 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\lambda = 1 > 0$ and $\lambda = -1 < 0$, so $(0,1)$ is a **saddle**

Case 3: $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\nabla F\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{bmatrix} -1+1 & 1 \\ -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = (-\lambda)(-\lambda) - (1)(-1) = \lambda^2 + 1$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Since the eigenvalues are purely imaginary (real part $= 0$), $\left(\frac{1}{2}, \frac{1}{2}\right)$ is neither stable, unstable, or a saddle

Case 4: $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$$\nabla F\left(-\frac{1}{2}, \frac{1}{2}\right) = \begin{bmatrix} -1+1 & -1 \\ 1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = (-\lambda)(-\lambda) - (1)(-1) = \lambda^2 + 1$$

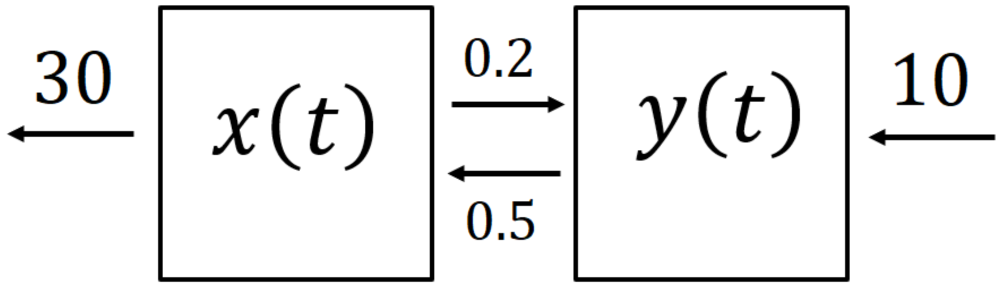
$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Once again $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is neither stable, unstable, or a saddle

STEP 3: Answer:

$$(0,0) \text{ saddle}$$
$$(0,1) \text{ saddle}$$
$$\left(\frac{1}{2}, \frac{1}{2}\right) \text{ neither}$$
$$\left(-\frac{1}{2}, \frac{1}{2}\right) \text{ neither}$$

6.



$$\begin{cases} x'(t) = -\,0.2x(t) + 0.5y(t) - 30 \\ y'(t) = 0.2x(t) - 0.5y(t) + 10 \end{cases}$$

Which you can write as $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ where

$$A = \begin{bmatrix} -0.2 & 0.5 \\ 0.2 & -0.5 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} -30 \\ 10 \end{bmatrix}$$