ANALYSIS – HOMEWORK 1

Problem 1: Show that $\mathbb{Z}$ (with the usual metric) is complete

Problem 2: Let (X, d) be a metric space and suppose (x_n) in X is a sequence with the property that every subsequence has a further subsequence that converges to x. Prove that (x_n) converges to x.

Problem 3:

(a) Suppose (x_n) is a bounded sequence in $\mathbb{R}$ with the property that every *convergent* subsequence of (x_n) converges to x. Show that (x_n) converges to x.

(b) Show that (a) is false if (x_n) is not bounded

Problem 4: Draw a picture of the unit ball $B((0,0),1)$ (the open ball centered at $(0,0)$ and radius 1) of each of the following metric spaces:

(a) $(\mathbb{R}^2, d_2)$ where $d_2(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$

(b) $(\mathbb{R}^2, d_1)$ where $d_1(\mathbf{x}, \mathbf{y}) = |y_1 - x_1| + |y_2 - x_2|$

(c) $(\mathbb{R}^2, d_\infty)$ where $d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|y_1 - x_1|, |y_2 - x_2|\}$

Problem 5: Find a metric space (X, d) for which the boundary $\partial B(x, r)$ of $B(x, r)$ is **NOT** equal to the sphere of radius r at x, that is $\{y \in S \mid d(x, y) = r\}$

Definition:

(1) If (X, d) is a metric space, then we say E is **dense** in X if $\overline{E} = X$

(2) (X, d) is **separable** if there is a countable subset E of X that is dense in X

Problem 6: Show that $\mathbb{R}^k$ is separable

Problem 7: (more challenging) Show that if (X, d) is a complete metric space and $\{U_n\}_{n=1}^{\infty}$ is a countable collection of open dense subsets of X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

Problem 8: Show that if (X, d) is a metric space and $f : [0, \infty) \to [0, \infty)$ is an increasing concave function such that

$$f(x) = 0 \Leftrightarrow x = 0$$

Then $f(d(x, y))$ is also a metric

Problem 9: Show that if (X_k, d_k) is a family of metric spaces, then the following is a metric on the product space $\Pi_{k=1}^{\infty} X_k$

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)} \right)$$

Problem 10: Let $a > 0$ and $b > \sqrt{a}$ be given. Define a sequence (s_n) by $s_1 = b$ and

$$s_{n+1} = \frac{1}{2} \left(s_n + \frac{a}{s_n} \right)$$

Show that (s_n) is a decreasing sequence that converges to $\sqrt{a}$.