HOMEWORK 2 – SOLUTIONS

Problem 1:

(a)
$$\begin{aligned} x \in (g \circ f)^{-1}(U) &\Leftrightarrow (g \circ f)(x) \in U \\ &\Leftrightarrow g(f(x)) \in U \\ &\Leftrightarrow f(x) \in g^{-1}(U) \\ &\Leftrightarrow x \in f^{-1}\left(g^{-1}(U)\right) \end{aligned}$$

(b) Suppose U is open, then since g is continuous, $g^{-1}(U)$ is open, and hence, since f is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is open, and therefore

$$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right) \text{ is open } \checkmark$$

Hence $g \circ f$ is continuous □

Problem 2: Let $\epsilon > 0$ be given, let $\delta = \frac{1}{2}$, then if $d(x, x_0) < \delta = \frac{1}{2} < 1$, then $x = x_0$, and therefore

$$d'(f(x), f(x_0)) = d'(f(x_0), f(x_0)) = 0 < \epsilon \checkmark$$

Hence any f is continuous □

Problem 3: Suppose E is path-connected but not connected. Since E is not connected, there are A and B, nonempty, open, and disjoint such that $A \cup B = E$.

Since A and B are nonempty, there is $a \in A$ and $b \in B$.

Since γ is path-connected, there is a path $\gamma : [0,1] \to E$ with $\gamma(0) = a$ and $\gamma(1) = b$

Now consider $A' = \gamma^{-1}(A)$ and $B' = \gamma^{-1}(B)$. Then since A and B are open and γ is continuous, we get A' and B' are open.

Moreover $0 \in A'$ since $\gamma(0) = a \in A$ and therefore A' is nonempty, and similarly B' is nonempty, and finally

$$\begin{aligned} A' \cap B' &= \gamma^{-1}(A' \cap B') = \gamma^{-1}(A') \cap \gamma^{-1}(B') = A \cap B = \emptyset \\ A' \cup B' &= \gamma^{-1}(A' \cup B') = \gamma^{-1}(A') \cup \gamma^{-1}(B') = A \cup B = [0,1] \end{aligned}$$

But therefore A' and B' are disjoint, open, nonempty subsets of $[0,1]$ whose union in $[0,1]$, which contradicts that $[0,1]$ is connected $\Rightarrow\Leftarrow$.

Hence E must be connected

For $\mathbb{R}$, let $a, b \in \mathbb{R}$ and consider the path $\gamma(t) = (1-t)a + tb$, which is continuous and has values in $\mathbb{R}$ and $\gamma(0) = a$ and $\gamma(1) = b$ ✓

Problem 4: Suppose not, then there is c such that $f(x) \neq c$ for all $x \in [a,b]$. This means that for all x, either $f(x) > c$ or $f(x) < c$, and therefore $[a,b] = A \cup B$ where

$$\begin{aligned} A &= \{x \in [a,b] \mid f(x) < c\} = f^{-1}((-\infty, c)) \\ B &= \{x \in [a,b] \mid f(x) > c\} = f^{-1}((c, \infty)) \end{aligned}$$

Now $A \cup B = \emptyset$ and A and B are nonempty since either $f(a)$ or $f(b)$ are in A or B

Moreover, A and B are open since f is continuous and $(-\infty, c)$ and (c, ∞) are open.
And therefore $[a, b] = A \cup B$ with A and B nonempty, open, and disjoint, which contradicts the fact that $[a, b]$ is connected. $\Rightarrow\Leftarrow$ $\Box$

Problem 5: Suppose (I_n) is a decreasing sequence of nonempty, closed, and bounded subsets of $\mathbb{R}^n$ and let $I = \bigcap_{n=1}^{\infty} I_n$.

I **is closed:** This is because the intersection of any number of closed sets is closed

I **is bounded:** This is because, $I \subseteq I_1$ and I_1 is bounded by assumption.

F **is nonempty:** For each $n = 1, 2, \dots,$ I_n is nonempty, so let x_n be an element of I_n.

Consider the sequence (x_n). Since for all n, $x_n \in I_n \subseteq I_1$, $x_n \in I_1$ for all n, and since I_1 is bounded (by assumption), then the sequence (x_n) is bounded (in $\mathbb{R}^n$).

Therefore, by the **Bolzano-Weierstrass** (x_n) has a subsequence (x_{n_k}) that converges to some $x \in \mathbb{R}^n$.

Claim: x is in I

Note: Then we would be done because, since $x \in I$, I is nonempty.

To show $x \in I$, we must show that for all $n_0 \in \mathbb{N}$, $x \in I_{n_0}$.

Let n_0 be arbitrary.

Then, for any $k \geq n_0$, we have $n_k \geq n_0$

Therefore, $I_{n_k} \subseteq I_{n_0}$ (since sets I_n are decreasing).

Therefore, for every $k \geq n_0$ $x_{n_k} \in I_{n_k}$ (by definition) $\subseteq I_{n_0}$ and so $x_{n_k} \in I_{n_0}$.

But then this means that all the terms of the sequence (x_{n_k}) are eventually in I_{n_0}

Therefore, since I_0 is closed (by assumption) the limit x of (x_{n_k}) is also in I_{n_0}, hence $x \in I_{n_0}$.

Hence, since n_0 was arbitrary we get $x \in I$, so I is nonempty. ✓ □

Problem 6: STEP 1: Suppose $[0, 1]$ can be written as the disjoint union of $[a_k, b_k]$ where $k \in \mathbb{N}$.

Let $x_0 = b_1$ and assume $b_1 < 1$ (since the union is countably infinite and the intervals are disjoint).

Then for some $0 < \epsilon_1 < \frac{1}{2}$, $x_0 + \epsilon_1 \in (0, 1)$, so $x_0 + \epsilon_1$ lies in some $[a_k, b_k]$. Let $x_1 = a_k > x_0$.

Then

$$|x_1 - x_0| = x_1 - x_0 = a_k - x_0 \leq x_0 + \epsilon_1 - x_0 = \epsilon_1 < \frac{1}{2}$$

Continuing, for some $\epsilon_2 < \frac{1}{4}$, $x_1 - \epsilon_2 \in (0,1)$ lies in some $[a_l, b_l]$, then let $x_2 = b_l$. So $x_0 < x_1 < x_2$ and $|x_1 - x_2| < \frac{1}{4}$.

STEP 3: Now suppose we have found $x_0, x_1, \ldots, x_{2k-1}, x_{2k}$ with

$$x_0 < x_2 < \cdots < x_{2k} < \cdots < x_{2k-1} < \cdots < x_1$$

And $|x_{2k+1} - x_{2k}| < \frac{1}{2^k}$

Then for some $\epsilon_{2k+1} < \frac{1}{2^{k+1}}$, $x_{2k} + \epsilon_{2k+1} \in (0,1)$, so $x_{2k} + \epsilon_{2k+1}$ lies in some $[a_l, b_l]$. Let $x_{2k+1} = a_l > x_{2k}$ and we can choose ϵ_{2k+1} small enough so that $x_{2k+1} < x_{2k-1}$.

And for some $\epsilon_{2k+2} < \frac{1}{2^{2k+2}}$, $x_{2k+2} - \epsilon_{2k+2} \in (0,1)$ lies in some $[a_p, b_p]$, then let $x_{2k+2} = b_p$. So $x_{2k+2} > x_{2k}$ and we can choose ϵ_{2k+2} small enough with $x_{2k+2} < x_{2k+1}$ ✓

STEP 4: Now consider the nested closed intervals $[x_{2n}, x_{2n+1}]$. By the Finite Intersection property $\cap_{n=1}^{\infty}[x_{2n}, x_{2n+1}]$ is nonempty, so there is $x \in [x_{2n}, x_{2n+1}]$. Because $|x_{2n+1} - x_{2n}| < 2^{-n}$, one can show that, as in problem 10.6, that (x_n) converges to x.

STEP 5:

Claim: $x \notin [0,1]$

Proof: Suppose $x \in [0,1] = \bigcup_{n=1}^{\infty}[a_n, b_n]$, then $x \in [a_p, b_p]$ for some p

If $x > 0$, then, since the subsequence (x_{2k}) is monotonically increasing and converges to x, there must be some n with $a_p < x_{2n} < x$. But by construction, x_{2k} is the right point of a sub-interval, so $x_{2k} = b_l$ for some l, which contradict the fact that $x \in [a_p, b_p]$ and the intervals are disjoint. $\Rightarrow\Leftarrow$, so there are no b_l between a_p and b_p.

Hence $x = 0$ but this contradicts $x > x_0 \geq 0$. $\square$

Problem 7: Using the definition of totally bounded with $r = \frac{1}{k}$ (for $k \in \mathbb{N}$), for all k, we can cover E with finitely many balls $B\left(x_1^k, \frac{1}{k}\right), B\left(x_2^k, \frac{1}{k}\right), \dots B\left(\right.$

Let F be the union of the centers $\left\{x_1^k, \dots, x_{n_k}^k\right\}$ as k ranges over all the integers. Then F is countable, being the countable union of finitely many sets.

Claim: F is dense in E

Let $x \in E$ and $r > 0$, we need to find $y \in F$ such that $y \in B(x, r)$. But if r is given, choose k large enough such that $\frac{1}{k} < r$. Then, since the balls $B\left(x_1^k, \frac{1}{k}\right), B\left(x_2^k, \frac{1}{k}\right), \dots B\left(x_{n_k}^k\right)$ cover E, we must have $x \in B\left(x_j^k, \frac{1}{k}\right)$ for some j. But then, by definition, $y =: x_j^k \in F$ (it's the center of a ball), but also $d(x, y) < \frac{1}{k} < r$, so $y \in B(x, r)$ ✓ $\square$

Problem 8:

F bounded: Notice that:

$$\begin{aligned} F &= \{x \mid \sup\{|x_j|, j = 1, 2, \dots\} \leq 1\} \\ &= \{x \mid \sup\{|x_j - 0|, j = 1, 2, \dots\} \leq 1\} \\ &= \{x \mid d(x, \mathbf{0}) \leq 1\} \\ &= \overline{B(\mathbf{0}, 1)} \end{aligned}$$

Hence F is included in the (closed) ball of center $\mathbf{0} = (0, 0, 0, \dots)$ and radius 1, and hence is bounded. ✓

F closed: Suppose $(x^(n))$ is a sequence in F that converges to x. Want to show $x \in F$.

But since each $x^{(n)}$ is in F, we have:

$$\sup\left\{\left|x_j^{(n)}\right|, j = 1, 2, \dots, \right\} \leq 1$$

So for all n and all j, $\left|x_j^{(n)}\right| \leq 1$

But now, letting n go to infinity and using $x^(n) \to x$, we get $|x_j| \leq 1$, where $x = (x_1, x_2, \dots)$ And since this is true for all j, we have:

$$\sup\{|x_j|, j = 1, 2, \dots\} \leq 1$$

F not compact:

STEP 1: Suppose, for sake of contradiction, that F is compact.

For each $x \in F$, let

$$U(x) = B(x, 1) = \{y \in B \mid d(y, x) < 1\}$$

STEP 2: For each $n \in \mathbb{N}$, let $x^{(n)}$ be defined by:

$$x_j^{(n)} = \begin{cases} -1 & \text{if } j = n \\ 1 & \text{if } j \neq n \end{cases}$$

Then for each j, $\left|x_j^{(n)}\right| = 1 \leq 1$, so $\sup\left\{\left|x_j^{(n)}\right|, j = 1, 2, \dots\right\} \leq 1$ and so $(x^{(n)}) \in F$

STEP 3:

Claim: If $x \in F$, then $m \neq n$ then $x^{(m)}$ and $x^{(n)}$ cannot both belong to $U(x)$

Proof: Suppose $x^{(m)}$ and $x^{(n)}$ both belong to $U(x)$, then by definition of $U(x)$, we have $d(x^{(m)}, x) < 1$ and $(x^{(m)}, x) < 1$, so

$$d(x^{(m)}, x^{(n)}) \leq d(x^{(m)}, x) + d(x^{(n)}, x) < 1 + 1 = 2$$

That is

$$\sup \left\{ \left| x_j^{(m)} - x_j^{(n)} \right|, j = 1, 2, \dots \right\} < 2$$

So for all j, $\left| x_j^{(m)} - x_j^{(n)} \right| < 2$

But if you let $j = n$, then you get (since $m \neq n$)

$$\left| x_n^{(m)} - x_n^{(n)} \right| = |1 - (-1)| = 2 < 2$$

Which is a contradiction ✓

STEP 4: Define:

$$\mathcal{U} = \{U(x) \mid x \in F\}$$

Then $\mathcal{U}$ is an open cover of F and therefore has a finite sub-cover $\mathcal{V} = \{U(x_1), \dots, U(x_n)\}$.

Now since $x^{(1)} \in F$, then, since $\mathcal{V}$ is a sub-cover, we must have $x^{(1)} \in V$ for some $V \in \mathcal{V}$, and WLOG assume $V = U(x_1)$, hence $x^{(1)} \in U(x_1)$.

Similarly $x^{(2)} \in V$ for some $V \in \mathcal{V}$. However by the above claim, $x^{(2)}$ cannot be in $U(x_1)$ since $x^{(1)}$ is already in $U(x_1)$, so $x^{(2)}$ must be in some different element of $\mathcal{V}$, so WLOG, assume $x^{(2)} \in U(x_2)$.

Continuing in this manner, we get that $x^{(k)} \in U(x_k)$ for $k = 1, \dots, n$, but then $x^{(n+1)}$ cannot be in $\mathcal{V}$ since it cannot be in any of the $U(x_1), \dots, U(x_n)$ since $n + 1 \neq 1, 2, \dots, n$ but this contradicts that $\mathcal{V}$ covers $E \Rightarrow\Leftarrow$

Problem 9: Consider $f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \tan^{-1}\left(\pi x - \frac{\pi}{2}\right)$$

Then, one can check that $g(x) = \pi x - \frac{\pi}{2}$ is continuous, one-to-one, and onto, and its inverse is continuous and therefore a homeomorphism.

Also since $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is continuous and one-to-one and onto $\mathbb{R}$ (you can show this using the fact that $\tan(x) \to \pm\infty$ near $\pm\frac{\pi}{2}$ and an analog of the Intermediate Value Theorem), its inverse $\tan^{-1}$ is continuous, and therefore a homeomorphism

Hence $f(x)$ is a homeomorphism, being a composition of two homeomorphisms, and therefore $(0, 1)$ and $\mathbb{R}$ are homeomorphic.

Problem 10:

(a) Since f is continuous, one-to-one, and onto its image, it suffices to show that f^{-1} is continuous.

> **Claim:** f is continuous if and only if for each closed set C, $f^{-1}(C)$ is closed

This follows because if f is continuous and C is closed, then C^c is open, and therefore $f^{-1}(C^c)$ is open, hence $\left(f^{-1}(C)\right)^c$ is open, so $f^{-1}(C)$ is closed ✓

Conversely, if $f^{-1}(C)$ is closed whenever C is closed, then if U is any open set, then U^c is closed, so by assumption $f^{-1}(U^c)$ is closed, and therefore $\left(f^{-1}(U)\right)^c$ is closed, and so $f^{-1}(U)$ is open, so f is continuous ✓

Now suppose C is an arbitrary closed subset of K, then since K is compact, C is a closed subset of a compact set, and hence compact. Therefore, since C is compact and f is continuous, $f(C)$ is compact, and hence closed.

Therefore, whenever C is closed, $f(C)$ is closed, and by the claim below, it follows that $\left(f^{-1}\right)^{-1}(C) = f(C)$ is closed, and so f^{-1} is continuous since f was arbitrary

Claim: $\left(f^{-1}\right)^{-1}(C) = f(C)$

Proof:

$$\begin{aligned} x \in \left(f^{-1}\right)^{-1}(C) &\Leftrightarrow f^{-1}(x) \in C \\ &\Leftrightarrow f\left(f^{-1}(x)\right) \in f(C) \\ &\Leftrightarrow x \in f(C) \checkmark \quad \square \end{aligned}$$

(b) Let

$$(x_n) = \left(\cos\left(2\pi - \frac{1}{n}\right), \sin\left(2\pi - \frac{1}{n}\right)\right)$$

Then (x_n) converges to $(1,0)$, but $f^{-1}(x_n) = 2\pi - \frac{1}{n}$ converges to $2\pi \neq f^{-1}((1,0)) = 0$.

Hence f^{-1} is not continuous.