ANALYSIS – HOMEWORK 2

Problem 1:

(a) If f and g are any functions, prove that

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

(b) Deduce that if f and g are continuous, then $g \circ f$ is continuous

Problem 2: Let (S, d) be $\mathbb{R}$ equipped with the discrete metric

$$d(x, y) = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \neq y \end{cases}$$

And let (S', d') be any metric space. Show that *any* function $f : S \to S'$ must be continuous

Problem 3:

Definition:

Let E be any subset of $\mathbb{R}$ (or of any metric space)

(1) A **path** in E is a continuous function $\gamma : [0, 1] \to E$

(2) E is **path-connected** if for any pair of points a and b in E, there is a path γ with $\gamma(0) = a$ and $\gamma(1) = b$

Show that if E is path-connected, then it is connected and deduce that $\mathbb{R}$ is connected

Problem 4: Prove the Intermediate Value Theorem: If f is continuous on $[a, b]$ then for every c (strictly) between $f(a)$ and $f(b)$ there exists $x \in (a, b)$ such that $f(x) = c$

Problem 5: Prove the Finite Intersection Property: If $I_1 \subseteq I_2 \subseteq \cdots$ is a decreasing sequence of nonempty, closed, and bounded subsets of $\mathbb{R}^n$ then $\bigcap_{n=1}^{\infty} I_n$ is closed, bounded, and nonempty.

Problem 6: Use the Finite Intersection Property to prove that $[0, 1]$ cannot be written as a countable infinite union of disjoint closed subintervals

Problem 7: Show that if E is totally bounded, then it is separable (= has a countable dense subset)

Problem 8: Let E be the set of all bounded sequences $\mathbf{x} = (x_1, x_2, \cdots)$ and define

$$d(\mathbf{x}, \mathbf{y}) = \sup \{|x_i - y_i| \mid i = 1, 2, \cdots\}$$

And let F consist of all $\mathbf{x} \in E$ such that $\sup \{|x_j| \mid i = 1, 2, \cdots\} \leq 1$

Show that F is closed and bounded, but not compact

Definition:

Let A and B be two subsets of $\mathbb{R}$ (or any two metric spaces) and $f : A \to B$ is a function, then:

(a) f is a **homeomorphism** if f is continuous, one-to-one, onto, and f^{-1} is continuous

(b) A and B are **homeomorphic** if there is a homemorphism between A and B

(c) A **topological property** is a property that is preserved under homeomorphisms

Problem 9: Find a homeomorphism between $(0, 1)$ and $\mathbb{R}$.

Problem 10:

(a) Show that if K is covering compact and $f : K \to f(K)$ is continuous and one-to-one, then f is a homeomorphism

(b) Let S^1 be the unit circle in $\mathbb{R}^2$. Consider the map $f : [0, 2\pi) \to S^1$ by $f(t) = (\cos(t), \sin(t))$. You may assume that f is continuous, one-to-one, and onto. Show that f^{-1} is not continuous and hence not a homeomorphism.