HOMEWORK 3 – SOLUTIONS

Problem 1: Suppose F is compact and $E \subseteq F$ is closed.

Let $\mathcal{U}$ be an open cover of E.

Then, since E is closed, E^c is open.

Now consider the cover

$$\mathcal{U}' =: \mathcal{U} \cup \{E^c\}$$

Claim: $\mathcal{U}'$ is an open cover of F

Proof: If $x \in F$ because either $x \in F$, in which case $x \in E$, so there is $U \in \mathcal{U} \subseteq \mathcal{U}'$ with $x \in U$. Or $x \notin E$, in which case $x \in E^c$ which is an element of $\mathcal{U}'$ ✓

But since F is compact, there is a finite sub-cover $\mathcal{V}'$ of $\mathcal{U}'$.

Since $E \subseteq F$, $\mathcal{V}'$ covers E.

But then $\mathcal{V} =: \mathcal{V}' \setminus \{E^c\}$. Then $\mathcal{V}$ is a subcover of $\mathcal{U}$ (since $E^c \notin \mathcal{U}$)

Claim: $\mathcal{V}$ covers E

Proof: Let $x \in E$, then since $\mathcal{V}'$ covers E, there must be $V \in \mathcal{V}'$ with $x \in V$. But $V \neq E^c$ because then we would get $x \in E^c$ which is a contradiction. Hence $V \in \mathcal{V}' \backslash \{E^c\} = \mathcal{V}$ and $x \in V$ ✓

Therefore we found a finite sub-cover $\mathcal{V}$ of $\mathcal{U}$ that covers E, and so E is compact $\square$

Problem 2:

STEP 1: Fix $x_0 \in \mathbb{R}^k$ and let $\epsilon > 0$ be given. Let $K_n = \overline{B(x_0, \frac{1}{n})}$, notice that the K_n are decreasing, and therefore, by (2), we have

$$\bigcap_{n=1}^{\infty} f(K_n) = f\left(\bigcap_{n=1}^{\infty} K_n\right) = f(\{x_0\}) = \{f(x_0)\}$$

STEP 2: Let $B = B(f(x_0), \epsilon) = (f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Then, first of all

$$\bigcap (f(K_n) \backslash B) = \left(\bigcap f(K_n)\right) \cap B^c = \{f(x_0)\} \backslash B = \emptyset$$

(because $f(x_0)$ is in B)

On the other hand, since K_n is compact, by (1), $f(K_n)$ is compact and hence closed, and so $f(K_n) \backslash B = f(K_n) \cap B^c$ is closed. And since the K_n are decreasing, the $f(K_n)$ are decreasing, and so is $f(K_n) \backslash B$.

Now if for all n, $(f(K_n) \backslash B) \neq \emptyset$, then by the finite intersection property we would have $\bigcap (f(K_n) \backslash B) \neq \emptyset$, which contradicts the above.

Therefore, for some N, $f(K_N) \backslash B = f(K_n) \cap B^c = \emptyset$.

STEP 3: But this implies that $f(K_N) \subseteq B$, and therefore, if $|x - x_0| < \frac{1}{N} \leq \frac{1}{N}$, then $x \in \overline{B(x_0, \frac{1}{N})} = K_N$, and so $f(x) \in f(K_N) \subseteq B = B(f(x_0), \epsilon)$, meaning $|f(x) - f(x_0)| < \epsilon$. In other words

$$|x - x_0| < \frac{1}{N} \Rightarrow |f(x) - f(x_0)| < \epsilon$$

STEP 4: Now given $\epsilon > 0$, let $\delta < \frac{1}{N}$ as above, then if $|x - x_0| < \delta < \frac{1}{N}$, then $|f(x) - f(x_0)| < \epsilon$, and therefore f is continuous at x_0, and hence is continuous. □

Problem 3: First fix any $0 < x \leq 1$. Then $f_n(x) = 0$ for all large enough n ($n \geq 2/x$ for example). Also $f_n(0) = 0$ for all n. Therefore

$$\lim_{n \to \infty} f_n(x) = 0 \qquad \text{for all } x \in [0, 1].$$

Note that f_n does not converge to 0 uniformly on $[0, 1]$.

Problem 4: Note that from

$$(\sqrt{n}|x| - 1)^2 \geq 0$$

it follows that

$$\frac{|x|}{1 + nx^2} \leq \frac{1}{2\sqrt{n}}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. From the inequality above it follows that f_n converges to $f(x) = 0$ uniformly on $\mathbb{R}$ as $n \to \infty$.

For the derivatives,

$$f_n'(x) = \frac{1 + nx^2 - x \cdot 2nx}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

Clearly, for $x \neq 0$ we have

$$f_n'(x) \to 0 = f'(x) \qquad \text{as } n \to \infty.$$

On the other hand, for $x = 0$ and all $n \in \mathbb{N}$ we have

$$f_n'(0) = 1.$$

Problem 5: ($\Rightarrow$) Let $\epsilon > 0$ be given. Then since $f_n \to f$ in $C[a, b]$, there is N such that if $n > N$ then $d(f_n, f) < \epsilon$, that is

$$\sup \{|f_n(x) - f(x)|, x \in [a, b]\} < \epsilon$$

(But if a sup is $< \epsilon$, then all its values are $< \epsilon$)

With the same N, $n > N$ then for all x, we have $|f_n(x) - f(x)| < \epsilon$, so $f_n \to f$ uniformly.

($\Leftarrow$) Similar □

Problem 6: For a function $h : E \to \mathbb{R}$ denote

$$\|h\| := \sup_{x \in E} |h(x)|.$$

Note that

$$h_n \to h \quad \text{uniformly on } E \iff \|h_n - h\| \to 0.$$

Uniform convergence $f_n + g_n \to f + g$ on E follows from

$$\|f_n + g_n - f - g\| \leq \|f_n - f\| + \|g_n - g\|.$$

To show that $f_n g_n \to fg$ uniformly on E

$$\|f_n g_n - fg\| = \|f_n g_n - f_n g + f_n g - fg\| \leq$$
$$\|f_n g_n - f_n g\| + \|f_n g - fg\| \leq \|f_n\| \|g_n - g\| + \|g\| \|f_n - f\|.$$

Claim: Suppose $h_n : E \to \mathbb{R}$ is a sequence of bounded functions such that (h_n) converges uniformly on E to some function h. Then there exists a number M that does not depend on x or n such that

$$|h_n(x)| \leq M \quad \text{and} \quad |h(x)| \leq M$$

for all $x \in E, n \in \mathbb{N}$.

Indeed, from the definition of uniform convergence with $\varepsilon = 1$ there exists $N \in \mathbb{N}$ such that

$$|h_n(x) - h(x)| \leq 1 \qquad \text{for all } n \geq N \text{ and } x \in E.$$

From the above it follows that

$$|h_n(x)| \leq |h_N(x)| + 2 \qquad \text{for all } n \geq N \text{ and } x \in E.$$

We can now take

$$M = \max\{\|h_1\|, \ldots, \|h_{N-1}\|, \|h_N\| + 2\}$$

which is a finite number.✓

From the claim above it follows that f_n and g_n are uniformly bounded on E. That is, there exists a number M such that

$$\|f_n\|, \|f\|, \|g_n\|, \|g\| \leq M$$

for all n. It therefore holds that

$$\|f_n g_n - fg\| \leq \|f_n\|\|g_n - g\| + \|g\|\|f_n - f\| \leq M\|g_n - g\| + M\|f_n - f\|$$

for all n. This implies the uniform convergence $f_n g_n \to fg$.

Problem 7: Choose any distinct points x and y in $\mathbb{R}$. Then by the mean value theorem, there exists a point c between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

Taking absolute values and rearranging, this becomes

$$|f(x) - f(y)| = |f'(c)||x - y| \leq L|x - y|,$$

which is the desired result.

Problem 8: Let $\epsilon > 0$, and choose $\delta = \epsilon/2L$. Then for all $x, y \in K$ with $|x - y| < \delta$ and for all $f \in A$,

$$|f(x) - f(y)| \leq L|x - y| \leq L\delta = \frac{\epsilon}{2} < \epsilon.$$