ANALYSIS – HOMEWORK 3

Problem 1: Use covering compactness to show that a closed subset of a compact set is compact.

Problem 2: (more challenging) Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ satisfies the following two conditions:

(1) For each compact set K, $f(K)$ is compact

(2) For any nested decreasing sequence of compact sets $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$, we have

$$f\left(\bigcap K_n\right) = \bigcap f(K_n)$$

Show that f is continuous

Problem 3: Consider the following sequence of functions f_n on $[0, 1]$, sometimes called the **growing steeple**

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}$$

Show that f_n converges pointwise to 0, but not uniformly to 0.

Problem 4: For $n = 1, 2, 3, \cdots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

Is true for $x \neq 0$ but false if $x = 0$

Problem 5: Show that $f_n \to f$ in $C[a, b]$ if and only if $f_n \to f$ uniformly

Problem 6: Let $\{f_n\}$ and $\{g_n\}$ be sequences of bounded functions such that $f_n \to f$ and $g_n \to g$ uniformly on a set E. Show that $f_n + g_n \to f + g$ and $f_n g_n \to fg$ uniformly on E.

Definition:

A function $f : (X, d) \to \mathbb{R}$ is **Lipschitz** if there exists a positive constant $L > 0$ such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq Ld(x, y)$$

Problem 7: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and $|f'(x)| \leq L$. Then f is Lipschitz continuous

Problem 8: Show that if $\{f_n\}$ is a family of Lipschitz functions with the same Lipschitz constant L, then $\{f_n\}$ is equicontinuous