HOMEWORK 4 – SOLUTIONS

Problem 1: Write $f_n(x) = g_n(x) + h_n(x)$ where $g_n(x) = \cos(x+n)$ and $h_n(x) = \ln(1 + \frac{\sin(nx)}{\sqrt{n+2}})$. It suffices to show that $\{g_n\}$ and $\{h_n\}$ are each equicontinuous on $[0, 2\pi]$.

Claim: $\{g_n\}$ is equicontinuous on $[0, 2\pi]$

Note $|g_n'(x)| = |\sin(x+n)| \leq 1$. Thus each g_n is Lipschitz continuous with Lipschitz constant 1, so $\{g_n\}$ is equicontinuous.

Claim: $\{h_n\}$ is equicontinuous on $[0, 2\pi]$.

It suffices to show h_n converges uniformly on $[0, 2\pi]$. We use the fact that if f is uniformly continuous and $g_n \to g$ uniformly, then $f \circ g_n \to f \circ g$ uniformly.

To see this fact, fix $\epsilon > 0$. Since f is uniformly continuous, we can choose δ so $|f(x) - f(y)|$ when $|x - y| < \delta$. Since $g_n \to g$ uniformly, we can choose N so $|g_n(x) - g(x)| < \delta$ when $n > N$. Then $|f(g(x)) - f(g_n(x))| < \epsilon$ when $n > N$, so $f \circ g_n \to f \circ g$ uniformly.

Now note $|\sin(nx)| \leq 1$, so $1 + \frac{\sin(nx)}{\sqrt{n+2}} \to 1$ uniformly on $[0, 2\pi]$ as $n \to \infty$. Note $1 + \frac{\sin(nx)}{\sqrt{n+2}}$ always lies within the compact set $[1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}]$ (again since $|\sin(nx)| \leq 1$), and ln is continuous hence uniformly continuous on compact sets. Thus by the above fact $h_n \to 0$ uniformly on $[0, 2\pi]$.

We conclude $\{h_n\}$ is equicontinuous on $[0, 2\pi]$.

Problem 2:

Claim: If $\{f_n\}$ is equicontinuous on $[0, 1]$ then f is constant on $[0, \infty)$.

Take any $\epsilon > 0$ and any $x, y \geq 0$. We will show $|f(x) - f(y)| < \epsilon$.

By equicontinuity, there exists δ such that if $a, b \in [0, 1]$ and $|a - b| < \delta$, then $|f_n(a) - f_n(b)| < \epsilon$ for all n.

Choose n large enough that $\frac{x}{n} \leq 1$, $\frac{y}{n} \leq 1$, and $|\frac{x}{n} - \frac{y}{n}| < \delta$. Then

$$|f(x) - f(y)| = \left| f_n\left(\frac{x}{n}\right) - f_n\left(\frac{y}{n}\right) \right| < \epsilon$$

Since ϵ is arbitrary, we conclude $f(x) = f(y)$, and thus f is constant.

Claim: Furthermore, if f is constant on $[0, \infty)$ then $\{f_n\}$ is equicontinuous on $[0, 1]$.

Given any $\epsilon > 0$, take $\delta = \frac{1}{2}$. Then for all $x, y \in [0, 1]$ with $|x - y| < \frac{1}{2}$, we have for all $n \in \mathbb{N}$ that $nx \geq 0$, $ny \geq 0$, and thus $f(nx) = f(ny)$. Thus $|f_n(x) - f_n(y)| = 0 < \epsilon$, so $\{f_n\}$ is equicontinuous on $[0, 1]$.

Problem 3: By Arzelà-Ascoli we only need to show that $\{F_n\}$ is bounded and equicontinuous.

Claim: $\{F_n\}$ is bounded.

Because $\{f_n\}$ is uniformly bounded, say $|f_n(x)| < M$, we have

$$|F_n(x)| = \left| \int_a^x f_n(t)dt \right| \leq \int_a^x |f_n(t)|dt \leq \int_a^b |f_n(t)|dt \leq M|a - b|.$$

So $\{F_n\}$ is bounded.

Claim: $\{F_n\}$ is equicontinuous.

Given $\epsilon > 0$ take $\delta = \frac{\epsilon}{M}$. Consider any $x > y$ with $|x - y| < \delta$. Then

$$|F_n(x)-F_n(y)| = \left|\int_a^x f_n(t)dt - \int_a^y f_n(t)dt\right| \leq \int_y^x |f_n(t)|dt \leq M|x-y| < M\delta = \epsilon.$$

Thus $\{F_n\}$ is equicontinuous.

Problem 4: Let $x, y \in \mathbb{R}$ with $x < y$. Split $[x, y]$ into n intervals of length $\frac{y-x}{n}$; the endpoints of these intervals are $x_k = x + \frac{k}{n}(y - x)$ for $k \in \{0, \cdots, n\}$. Then we have for some constant C

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \\ &\leq C \sum_{k=0}^{n-1} |x_{k+1} - x_k|^\alpha \\ &\leq C \sum_{k=0}^{n-1} \left(\frac{y-x}{n}\right)^\alpha \\ &= C|y - x|^\alpha n^{1-\alpha}. \end{aligned}$$

As $n \to \infty$, we have $C|y - x|^\alpha n^{1-\alpha} \to 0$, so $|f(x) - f(y)| = 0$. Thus f is constant.

Problem 5:

First ODE:

$$\begin{aligned}\frac{du}{dt} =& u^2\\ \frac{du}{u^2} =& dt\\ \int \frac{du}{u^2} =& \int dt\\ -\frac{1}{u} =& t + C\\ u =& -\frac{1}{t+C}\end{aligned}$$

$$u(0) = 1 \Rightarrow -\frac{1}{0+C} = 1 \Rightarrow -\frac{1}{C} = 1 \Rightarrow C = -1$$

$$u = -\left(\frac{1}{t-1}\right) = \frac{1}{1-t}$$

Second ODE:

$$\begin{aligned}\frac{du}{dt} =& \sqrt{u}\\ \frac{du}{\sqrt{u}} =& dt\\ \int \frac{du}{\sqrt{u}} =& \int dt\\ 2\sqrt{u} =& t + C\\ \sqrt{u} =& \frac{t+C}{2}\\ u =& \left(\frac{t+C}{2}\right)^2\end{aligned}$$

$$u(0) = \left(\frac{C}{2}\right)^2 = \frac{C^2}{4} = 0 \Rightarrow C = 0$$

$$u = \left(\frac{t}{2}\right)^2 = \frac{t^2}{4}$$

Problem 6:

Take $f(x) = \ln(e^x + 1)$. Then

$$|f'(x)| = \left|\frac{e^x}{e^x + 1}\right| < 1.$$

But if f has a fixed point x then $x = \ln(e^x + 1)$, so $e^x = e^x + 1$; this is impossible, so f has no fixed point.

Problem 7:

STEP 1: Let $x_0 \in X$ and define $x_n = f^n(x_0)$ (f applied n times)

Notice $d(x_1, x_2) = d(f(x_0), f(x_1)) \leq kd(x_0, x_1)$ and

$d(x_2, x_3) = d(f(x_1), f(x_2)) \leq kd(x_1, x_2) \leq kkd(x_0, x_1) = k^2 d(x_0, x_1)$

And more generally you can show that

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

STEP 2: Claim: (x_n) is Cauchy

Why? Let $\epsilon > 0$ be given and N be TBA, then if $m, n > N$ (WLOG assume $n \geq m$), then

$$
\begin{aligned}
d(x_m, x_n) \leq & d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\
\leq & k^m d(x_0, x_1) + k^{m+1} d(x_0, x_1) + \cdots + k^{n-1} d(x_1, x_0) \qquad \text{(By } \textbf{STEP 1}\text{)} \\
\leq & \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) d(x_1, x_0) \\
= & k^m \left(1 + k + \cdots + k^{n-m-1}\right) d(x_0, x_1) \\
\leq & k^m \left(1 + k + k^2 + \cdots\right) d(x_0, x_1) \\
= & k^m \left(\frac{1}{1-k}\right) d(x_0, x_1) \\
\leq & \frac{k^N}{1-k} d(x_0, x_1) \quad \text{Since } m > N \text{ and } k < 1
\end{aligned}
$$

But since $k < 1$ we have $\lim_{n\to\infty} k^n = 0$, so we can choose N large enough so that $\frac{k^N}{1-k} d(x_0, x_1) < \epsilon$, which in turn implies $d(x_m, x_n) < \epsilon$ ✓

STEP 3: Since (x_n) is Cauchy and X is complete, (x_n) converges to some p

Claim: p is a fixed point of f.

This follows because

$$
\begin{aligned}
x_{n+1} = & f(x_n) \\
\lim_{n\to\infty} x_{n+1} = & \lim_{n\to\infty} f(x_n) \\
p = & f\left(\lim_{n\to\infty} x_n\right) \qquad \text{(continuity)} \\
p = & f(p) \checkmark
\end{aligned}
$$

STEP 4: **Uniqueness:** Suppose there are two fixed points $p \neq q$, then

$$
d(p,q) = d(f(p), f(q)) \leq k d(p,q) < d(p,q) \Rightarrow\Leftarrow
$$