ANALYSIS – HOMEWORK 4

Problem 1: Show that the sequence (f_n) defined by

$$f_n(x) = \cos(x+n) + \ln\left(1 + \frac{\sin(nx)}{\sqrt{n+2}}\right)$$

is equicontinuous on $[0, 2\pi]$

Problem 2: Suppose f is a real continuous function on $\mathbb{R}$ and let

$$f_n(x) = f(nx) \text{ for } n = 1, 2, 3, \cdots$$

Suppose $\{f_n\}$ is equicontinuous on $[0, 1]$. What can you say about f?

Problem 3: Let (f_n) be a uniformly bounded sequence of Riemann integrable functions on $[a, b]$ and put

$$F_n(x) = \int_a^x f_n(t)dt \qquad (a \leq x \leq b)$$

Show that there is a subsequence (F_{n_k}) which conv uniformly on $[a, b]$

Problem 4: Show that if $f : \mathbb{R} \to \mathbb{R}$ is Hölder continuous with Hölder constant $\alpha > 1$ then f is constant

Problem 5: Use separation of variables to find solutions to

$$\begin{cases} \dfrac{du}{dt} = u^2 \\ u(0) = 1 \end{cases} \qquad\qquad \begin{cases} \dfrac{du}{dt} = \sqrt{u} \\ u(0) = 0 \end{cases}$$

Definition:

f has a **fixed point** if $f(p) = p$ for some p

Problem 6: Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f'(x)| < 1$ for all x but f has no fixed point.

Definition:

If (X, d) is a metric space, then $f : X \to X$ is a contraction if there is $k < 1$ such that for all x and y we have

$$d(f(x), f(y)) \leq kd(x, y)$$

Problem 7: Prove the **Banach Fixed Point Theorem:** If (X, d) is complete and $f : X \to X$ is a contraction, then f has a unique fixed point p

Hint: Let $x_0 \in X$ and consider $x_n = f^n(x_0)$ and show (x_n) is Cauchy