HOMEWORK 5 – SOLUTIONS

Problem 1: To prove the claim we use Arzela-Ascoli theorem. To use the theorem we need to check equicontinuity and uniform boundedness.

Equicontinuity immediately follows from the uniform boundedness of (f_n'):

$$|f_n'| \leq M \Rightarrow |f_n(x) - f_n(y)| \leq M|x - y|$$

for all n and $x \in [a, b]$.

Uniform boundedness:

Since $|f_n(x_0)| \leq C$ for all n and some $0 \leq C < \infty$, then

$$|f_n(x)| \leq |f_n(x_0)| + |\int_{x_0}^{x} f_n'(s)ds| \leq C + M|x - x_0| \leq C + M(b - a)$$

for all $n \in \mathbb{N}$ and $x \in [a, b]$.

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz with Lipschitz constant L. Suppose $y_1, y_2 : [t_1, t_2] \to \mathbb{R}$ are both solutions of

$$y' = f(y)$$

with

$$y_1(0) = y_2(0) = y_0,$$

where $-\infty < t_1 < 0 < t_2 < \infty$.

Let $z(t) = (y_1(t) - y_2(t))^2$. Then

$$|z'(t)| = |2(y_1(t) - y_2(t))(y_1'(t) - y_2'(t))| =$$
$$= 2|y_1(t) - y_2(t)||f(y_1(t)) - f(y_2(t))| \leq [f \text{ Lipschitz}] \leq$$
$$2L|y_1(t) - y_2(t)|^2 = 2Lz(t).$$

Then for $t \in [0, t_2]$ it holds that

$$z(t) = \int_0^t z'(s)ds \leq \int_0^t 2Lz(s)ds.$$

Gronwall's inequality with $C = 0$ and $g(t) = 2L$ gives

$$z(t) = 0 \qquad \text{for all } t \in [0, t_2].$$

Similarly, applying Gronwall's inequality to $w(t) = z(-t)$ for $t \in [0, -t_1]$ gives

$$z(t) = 0 \qquad \text{for all } t \in [t_1, 0].$$

From the definition of $z(t) = (y_1(t) - y_2(t))^2$ we get

$$y_1(t) = y_2(t)$$

for all $t \in [t_1, t_2]$.

Problem 3: Here $g(t) \geq 0$, $u(t)$, $C(t)$ are continuous functions on $[a, b]$.

(a) Suppose $u(t)$ satisfies

$$u(t) \leq C(t) + \int_a^t g(s)u(s)ds.$$

Introduce $v(t) = u(t) - C(t)$. Then in terms of v the inequality above reads as

$$v(t) \leq \int_a^t g(s)(v(s) + C(s))ds.$$

Let

$$w(t) = \int_a^t g(s)(v(s) + C(s))ds.$$

Then $w(t)$ is continuously differentiable on $[a,b]$ with

$$w'(t) = g(t)(v(t) + C(t)) \leq g(t)w(t) + g(t)C(t).$$

Equivalently,

$$w'(t) - g(t)w(t) \leq g(t)C(t).$$

Multiply the inequality by the so-called integrating factor $e^{-\int_a^t g(s)ds}$ to get

$$w'(t)e^{-\int_a^t g(s)ds} - g(t)e^{-\int_a^t g(s)ds}w(t) \leq g(t)C(t)e^{-\int_a^t g(s)ds}.$$

The left side of the inequality is now a total derivative:

$$\frac{d}{dt}\Big(w(t)e^{-\int_a^t g(s)ds}\Big) \leq g(t)C(t)e^{-\int_a^t g(s)ds}.$$

Integrate the above inequality over $[a,t]$ to get (note that $w(a) = 0$)

$$w(t)e^{-\int_a^t g(s)ds} \leq \int_a^t g(s)C(s)e^{-\int_a^s g(r)dr}ds$$

Multiplying the above inequality by $e^{\int_a^t g(s)ds}$ and using $\int_a^t - \int_a^s = \int_s^t$ gives

$$w(t) \leq \int_a^t g(s)C(s)e^{\int_s^t g(r)dr}ds.$$

Now from

$$u(t) = v(t) + C(t) \leq w(t) + C(t)$$

we deduce

$$u(t) \leq C(t) + \int_a^t g(s)C(s)e^{\int_s^t g(r)dr}ds.$$

(b) Now suppose $C(t)$ is increasing. Then

$$\int_a^t g(s)C(s)e^{\int_s^t g(r)dr}ds \le C(t)\int_a^t g(s)e^{\int_s^t g(r)dr}ds.$$

Note that

$$\int_a^t g(s)e^{\int_s^t g(r)dr}ds = -\int_a^t \frac{d}{ds}\Big(e^{\int_s^t g(r)dr}\Big)ds =$$
$$= -e^{\int_s^t g(r)dr}|_a^t = e^{\int_a^t g(r)dr} - 1.$$

From part (a) we now obtain

$$u(t) \le C(t) + \int_a^t g(s)C(s)e^{\int_s^t g(r)dr}ds \le$$
$$\le C(t)(1 + \int_a^t g(s)e^{\int_s^t g(r)dr}ds) = C(t)e^{\int_a^t g(r)dr}.$$

Problem 4: Suppose $y(t)$ is a solution of $y'(t) = f(y(t))$, defined for $t \in [0, T)$ for some $0 < T < \infty$.

(a) Suppose $|f(x)| \le M$ for all $x \in \mathbb{R}$. Then

$$|y'(t)| = |f(y(t))| \le M$$

and therefore

$$|y(t)| \le |y(0)| + Mt \le |y(0)| + MT.$$

Therefore $y(t)$ remains bounded when t stays bounded.

(b) Now suppose f grows at most linearly. That is, $|f(x)| \le C|x| + K$ for all $x \in \mathbb{R}$. Then

$$|y(t) - y(0)| = |\int_0^t y'(s)| = |\int_0^t f(y(s))| \le$$
$$\int_0^t (K + C|y(s)|)ds \le KT + C\int_0^t |y(s)|ds.$$

Consequently,

$$|y(t)| \leq |y(0)| + KT + C\int_0^t |y(s)|ds$$

for all $t \in [0, T)$. Applying Gronwall's inequality to $|y(t)|$ gives

$$|y(t)| \leq (|y(0)| + KT)e^{Ct} \leq (|y(0)| + KT)e^{CT}.$$

Again, $y(t)$ remains bounded when t stays bounded.

Problem 5:

STEP 1: Main Observation: By integrating the ODE, it is equivalent to

$$\begin{aligned}\int_0^t y'(s)ds &= \int_0^t f(y(s))ds \\ y(t) - y_0 &= \int_0^t f(y(s))ds \\ y(t) &= y_0 + \int_0^t f(y(s))ds\end{aligned}$$

STEP 2: Let $\tau > 0$ TBA

Since f is continuous, it is bounded around y_0: There is some $r > 0$ and $C > 0$ such that $|f(x)| \leq C$ for all $x \in [y_0 - r, y_0 + r]$.

Let X be the space of continuous functions $y : [-\tau, \tau] \to [y_0 - r, y_0 + r]$ with the sup norm.

Given $y \in X$, define $\Phi(y) \in X$ (to be shown) by

$$\Phi(y)(t) = y_0 + \int_0^t f(y(s))ds$$

We're done once we show that Φ has a fixed point y, because then $\Phi(y) = y$ and we get

$$y(t) = y_0 + \int_0^t f(y(s))ds \checkmark$$

STEP 3: Proof that Φ is a contraction

First show that $\Phi : X \to X$: Notice that if y is continuous, then $\int_0^t f(y)$ is continuous (in fact differentiable) and hence $\Phi(y)(t)$ is continuous. Moreover

$$|\Phi(y)(t) - y_0| = \left|\int_0^t f(y(s))ds\right| \leq \int_0^t |f(y)|\, ds \leq \int_0^t C ds = Ct \leq C\tau \leq r$$

Provided you choose τ such that $\tau C \leq r$

Hence $\Phi(y) \in [y_0 - r, y_0 + r]$ and so $\Phi(y) \in X$.

Moreover, Φ is a contraction because

$$\begin{aligned}
d(\Phi(y),\Phi(z)) =& \sup_t \left| y_0 + \int_0^t f(y(s))ds - \left(y_0 + \int_0^t f(z(s))ds \right) \right| \\
\leq& \sup_t \left| \int_0^t f(y(s)) - f(z(s))ds \right| \\
\leq& \sup_t \int_0^t \left| f(y(s)) - f(z(s)) \right| ds \\
\leq& \int_0^\tau \left| f(y(s)) - f(z(s)) \right| ds \text{ (the integral is increasing in } t) \\
\leq& \int_0^\tau \left(\sup_s \left| f(y(s)) - f(z(s)) \right| \right) ds \\
=& \left(\sup_s \left| f(y(s)) - f(z(s)) \right| \right) \int_0^\tau 1 \\
\leq& L \sup_s \left| y(s) - z(s) \right| \tau \\
=& L\tau d(y,z)
\end{aligned}$$

This becomes a contraction provided we choose τ so that $L\tau < 1$

STEP 4: Uniqueness

Any other solution $z(t)$ is also a fixed point of Φ, that is $\Phi(z) = z$. Since a contraction has a unique fixed point, we have $z = y$. This is what local uniqueness means. □