ANALYSIS – HOMEWORK 5

Problem 1: Let $f_n[a,b] \to \mathbb{R}$ be a sequence of smooth functions such that (f_n') is bounded. Moreover, suppose for some x_0, $(f_n(x_0))$ is a bounded sequence. Show that (f_n) has a subsequence which converges uniformly on $[a,b]$

Problem 2: Use Grönwall's inequality to directly show that if y_1 and y_2 solve the ODE

$$\begin{cases} y' &= f(y) \\ y(0) &= y_0 \end{cases}$$

Where f is Lipschitz, then $y_1 = y_2$

Hint: Consider $z(t) = (y_1(t) - y_2(t))^2$

Problem 3: Here $g(t) \geq 0$, $u(t)$ and $C(t)$ are cont functions on $[a,b]$

(a) Prove the following generalization of Grönwall's inequality: If $u(t)$ satisfies

$$u(t) \leq C(t) + \int_a^t g(s)u(s)ds$$

$$\text{Then } u(t) \leq C(t) + \int_a^t C(s)g(s)e^{\int_s^t g(r)dr}ds$$

(b) If in addition $C(t)$ is increasing, show that

$$u(t) \leq C(t)e^{\int_a^t g(s)ds}$$

Problem 4: Consider the ODE $y' = f(y)$ where $|f(x)| \leq M$

(a) Show that no solution of the ODE escapes to infinity in finite time

(b) Show the same thing is true if $|f(x)| \leq C\,|x| + K$

Recall the Banach Fixed Point Theorem that we proved last time

Banach Fixed Point Theorem:

If (X, d) is complete and $f : X \to X$ is a contraction, then f has a unique fixed point p

Problem 5: (More challenging)

Use the Banach Fixed Point Theorem to show that if f is Lipschitz and $y_0 \in \mathbb{R}$ then for some small $\tau > 0$ there is a sol $y : [-\tau, \tau] \to \mathbb{R}$ of

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases}$$

Hint: First, since f is continuous, there is $r > 0$ and $C > 0$ such that $|f(x)| \leq C$ on $[y_0 - r, y_0 + r]$. Now let X be the space of continuous functions $y : [-\tau, \tau] \to [y_0 - r, y_0 + r]$. And given $y \in X$, let

$$\Phi(y)(t) = y_0 + \int_0^t f(y(s))ds$$

Show that $\Phi : X \to X$ and that Φ is a contraction