HOMEWORK 6 – SOLUTIONS

Problem 1: Let $p_n(x) = x^n$, where $n \in \mathbb{N}$. Then $T(p_n) = p_n' = nx^{n-1}$. Note that

$$\|p_n\| = \sup_{x\in[0,1]} x^n = 1$$

and

$$\|T(p_n)\| = \sup_{x\in[0,1]} nx^{n-1} = n.$$

Therefore

$$\sup_{\|p\|=1} \frac{\|Tp\|}{\|p\|} = \infty.$$

That is, the operator T is unbounded on $(P, \|\cdot\|)$. Equivalently, there does not exist C such that

$$\|T(p)\| \le C\|p\|$$

for all $p \in P$.

Problem 2: Finite-dimensionality is crucial here!

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ be given. Then there are $c_1, \dots, c_n$ such that

$$x = c_1 e_1 + \cdots + c_n e_n$$

$$\text{Then } \; |Ax| = \left| A\left(\sum_{i=1}^n c_i e_i\right)\right| \overset{\text{LIN}}{=} \left|\sum_{i=1}^n c_i (Ae_i)\right| \le \sum_{i=1}^n |c_i|\,|Ae_i|$$

However, for each i, we have

$$|c_i| = \sqrt{(c_i)^2} \leq \sqrt{(c_1)^2 + \cdots + (c_n)^2} = |x|$$

$$\text{Hence } |Ax| \leq \sum_{i=1}^{n} \underbrace{|c_i|}_{\leq |x|} |Ae_i| \leq \left(\sum_{i=1}^{n} |Ae_i| \right) |x| = C\,|x|$$

Where $C =: \sum_{i=1}^{n} |Ae_i|$ (doesn't depend on x).

From this it follows that for all x, $\frac{|Ax|}{|x|} \leq C < \infty$ □

Problem 3: (a) follows because for all x

$$|(L+S)x| = |Lx + Sx| \leq |Lx| + |Sx| \leq \|L\|\,|x| + \|S\|\,|x| = (\|L\| + \|S\|)\,|x|$$

And (b) follows similarly because for all x

$$|(LS)x| = |L(Sx)| \leq \|L\|\,|Sx| \leq \|L\|\,\|S\|\,|x|$$

Problem 4: If

$$A(x_1, x_2) = (x_1, 2x_2)$$

Then can show that the operator $\|A\| = 2$ but

$$[A] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Problem 5: STEP 1: Suppose f has two derivatives A and B at x

Then for all $y \in \mathbb{R}^n$, we have

$$\begin{aligned} f(x+y) =& f(x) + Ay + r(y) \\ f(x+y) =& f(x) + By + s(y) \end{aligned}$$

Subtracting the second equation from the first, we get

$$Ay - By = s(y) - r(y)$$

In particular, this implies that

$$\lim_{y\to 0}\frac{|Ay-By|}{|y|} = \lim_{y\to 0}\frac{|s(y)-r(y)|}{|y|} \leq \lim_{y\to 0}\frac{|s(y)|+|r(y)|}{|y|} = 0$$

STEP 2: Notice that for any $t \in \mathbb{R}$, $\lim_{t\to 0} ty = 0$ and so by the above

$$0 = \lim_{t\to 0}\frac{|A(ty)-B(ty)|}{|ty|} \overset{\text{LIN}}{=} \lim_{t\to 0}\frac{|t|\,|Ay-By|}{|t|\,|y|} = \lim_{t\to 0}\underbrace{\frac{|Ay-By|}{|y|}}_{\text{Constant}} = \frac{|Ay-By|}{|y|}$$

So in fact for every y we have

$$\frac{|Ay-By|}{|y|} = 0 \Rightarrow |Ay-By| = 0 \Rightarrow Ay = By \Rightarrow A = B \quad \square$$

Problem 6: STEP 1: Fix x and let $A = f'(x)$ and $B = g'(f(x))$. Using the definition of F then the definition of f' and of g' we get

$$\begin{aligned} F(x+h) =& g(f(x+h)) = g(f(x) + Ah + r_f(h)) \\ =& g(f(x)) + B\left(Ah + r_f(h)\right) + r_g\left(Ah + r_f(h)\right) \end{aligned}$$

$$F(x+h) = F(x) + BAh + Br_f(h) + r_g\left(Ah + r_f(h)\right)$$

If we show that the remainder terms are sublinear, then we would be done because then

$$F'(x) = BA = g'(f(x))f'(x)$$

STEP 2: Remainder Terms

$$\frac{|Br_f(h)|}{|h|} \leq \|B\| \left(\frac{|r_f(h)|}{|h|} \right) \stackrel{h\to 0}{\to} 0$$

For the second term, first notice that

$$|Ah + r_f(h)| \leq \|A\| \, |h| + |r_f(h)| \stackrel{h\to 0}{\to} 0$$

Therefore, by definition of r_g we have

$$\frac{|r_g(Ah + r_f(h))|}{|h|} = \frac{|r_g(Ah + r_f(h))|}{|Ah + r_f(h)|} \times \frac{|Ah + r_f(h)|}{|h|} \stackrel{h\to 0}{\to} 0$$

This follows from the definition of r_g and because its input goes to 0, while the second term is bounded. □

Technical Note: It is possible that $Ah + r_f(h) = 0$, but this can be dealt with by redefining $r_g(0) = 0$ if necessary.

Problem 7: Fix j, then since f is differentiable at x, we have

$$f(x + te_j) - f(x) = f'(x)\,(te_j) + r(te_j)$$

Where $\lim_{t\to 0} \frac{|r(te_j)|}{te_j} = \lim_{t\to 0} \frac{|r(te_j)|}{|t|} = 0$

Dividing both sides by t and using linearity of $f'(x)$ we get

$$\lim_{t\to 0} \frac{f(x + te_j) - f(x)}{t} = f'(x)(e_j)$$

The left hand side is by definition $\begin{bmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{bmatrix}$ and right-hand-side is by definition the $j-$th column of $[f'(x)]$, so both columns are equal. Since this is true for all j, the two matrices are equal. □

Problem 8: For (a) we have

$$f(2\pi) - f(0) = (1,0) - (1,0) = (0,0) \neq f'(c)\,(2\pi - 0) \text{ for any } c$$

Since $f'(c) = (-\cos(c), \sin(c)) \neq (0,0)$

Proof of (b)

STEP 1: Fix a and b and define the segment from a to b:

$$\gamma(t) = (1-t)a + tb \qquad (0 \leq t \leq 1)$$

$$\text{Let } g(t) = f(\gamma(t)) \text{ (Collects } f \text{ along the segment)}$$
$$\text{Then } g'(t) = f'(\gamma(t))\gamma'(t) = f'(\gamma(t))(b-a) \text{ (Definition of } \gamma')$$
$$\text{Hence } \|g'(t)\| \leq \|f'(\gamma(t))(b-a)\| = \|f'(\gamma(t))\|\,|b-a| \leq M\,|b-a|$$

Notice the above is valid for all t.

Claim: There is some c with $|g(1) - g(0)| \leq \|g'(c)\|$

Then we would be done because

$$|f(b) - f(a)| = |f(\gamma(1)) - f(\gamma(0))| = |g(1) - g(0)| \overset{\text{Claim}}{\leq} \|g'(c)\| \leq M\,|b-a| \checkmark$$

STEP 2: Proof of Claim:

$$\text{Let } \phi(t) = (g(1) - g(0)) \cdot g(t) \text{ (Scalar function)}$$

Then by the single-variable MVT applied to ϕ there is c in $(0,1)$ with

$$\phi(1) - \phi(0) = \phi'(c) \overset{\text{DEF}}{=} (g(1) - g(0)) \cdot g'(c)$$

$$\begin{aligned} \text{But also } \phi(1)-\phi(0) &\overset{\text{DEF}}{=} (g(1)-g(0))\cdot g(1)-(g(1)-g(0))\cdot g(0) \\ &= (g(1)-g(0))\cdot(g(1)-g(0)) \\ &= |g(1)-g(0)|^2 \end{aligned}$$

Hence $|g(1)-g(0)|^2 = \phi(1)-\phi(0) = (g(1)-g(0))\cdot g'(c) \overset{\text{C-S}}{\leq} |g(1)-g(0)| \, \|g'(c)\|$
Dividing both sides by $|g(1)-g(0)|$ we get $|g(1)-g(0)| \leq \|g'(c)\|$ $\square$
If $f'(x)=0$ then setting $M=0$ in the above we get $f(b)-f(a)=0$ and so f is constant since a and b are arbitrary.

Problem 9: By the definition of differentiability,

$$\begin{aligned} f(x+h) &= f(x)+f'(x)h+r_f(h), \\ g(x+h) &= g(x)+g'(x)h+r_g(h), \end{aligned}$$

where

$$\frac{r_f(h)}{h} \to 0, \frac{r_g(h)}{h} \to 0 \qquad \text{as } h \to 0.$$

Then

$$\begin{aligned} f(x+h)g(x+h) = (f(x)+f'(x)h+r_f(h))(g(x)+g'(x)h+r_g(h)) = \\ = f(x)g(x)+(f'(x)g(x)+f(x)g'(x))h+r_{fg}(h), \end{aligned}$$

where

$$\begin{aligned} r_{fg}(h) := g(x)r_f(h)+f'(x)g'(x)h^2+g'(x)hr_f(h)+f(x)r_g(h)+ \\ +f'(x)hr_g(h)+r_f(h)r_g(h). \end{aligned}$$

Clearly,

$$\frac{r_{fg}(h)}{h} \to 0 \qquad \text{as } h \to 0.$$

Therefore fg is differentiable at x with

$$(fg)'(x) = f'(x)g(x)+f(x)g'(x).$$

Problem 10: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \text{for } (x,y) \neq (0,0).$$

Then for $(x,y) \neq 0$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2},$$
$$\frac{\partial f}{\partial y}(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

From $f(h,0) = f(0,h) = 0$ for all $h \in \mathbb{R}$ it follows that

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0,$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$$

Note that

$$f(h,h) = \frac{h^2}{h^2+h^2} = \frac{1}{2}$$

for all $h \neq 0$. Therefore

$$f(h,h) = \frac{1}{2} \not\to 0 = f(0,0) \qquad \text{as } h \to 0$$

and f is not continuous at $(x,y) = (0,0)$.

Problem 11: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x,y) = \frac{x^3}{x^2+y^2} \qquad \text{for } (x,y) \neq (0,0).$$

Then for $(x,y) \neq 0$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2},$$
$$\frac{\partial f}{\partial y}(x,y) = -\frac{2x^3y}{(x^2+y^2)^2},$$

and

$$0 \leq \frac{\partial f}{\partial x}(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \leq \frac{(x^2+y^2)(3x^2+3y^2)}{(x^2+y^2)^2} = 3,$$
$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \frac{|2xy|x^2}{(x^2+y^2)^2} \leq \frac{(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^2} = 1.$$

From $f(x,0) = x$ and $f(0,y) = 0$ it follows that

$$\frac{\partial f}{\partial x}(0,0) = 1,$$
$$\frac{\partial f}{\partial y}(0,0) = 0.$$

Therefore $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist on $\mathbb{R}^2$ and are bounded functions on $\mathbb{R}^2$.

If f is differentiable at $(x,y) = (0,0)$, then

$$f(x,y) = f(0,0) + \left[\frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y\right] + r(x,y),$$

where

$$\lim_{x,y\to 0} \frac{r(x,y)}{\sqrt{x^2+y^2}} = 0.$$

Here we have

$$r(x,y) = f(x,y) - x = -\frac{xy^2}{x^2+y^2}.$$

Take $x = y = h > 0$, then

$$\frac{r(h,h)}{\sqrt{2}h} = -\frac{h^3}{2h^2\sqrt{2}h} = -2^{-\frac{3}{2}} \neq 0.$$

Therefore f is not differentiable at $(x, y) = (0, 0)$.