ANALYSIS – HOMEWORK 6

Problem 1: Let P be the set of polynomials on $[0,1]$ with the supremum norm

$$\|p\| = \sup_{x\in[0,1]} |p(x)|$$

And let $T : P \to P$ be defined by $T(p) = p'$

Show that there is no C such that $\|T(p)\| \leq C\,\|p\|$. This is an example of an unbounded linear transformation.

Problem 2: Show that any linear transformation $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is bounded.

Problem 3: Show that if L and S are bounded then

(a) $\|L + S\| \leq \|L\| + \|S\|$

(b) $\|LS\| \leq \|L\|\,\|S\|$

Whenever those expressions are defined. Here LS is the composition of L and S, that is $LS(x) = L(S(x))$

Problem 4: Give an example of a 2×2 matrix A such that the operator norm of $L(x) = Ax$ is strictly less than the square root of the sum of squares of the entries of A

Problem 5: Show that the derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ is unique

Problem 6: Prove the Chain Rule: If f and g are differentiable and $F(x) = f(g(x))$ then

$$F'(x) = f'(g(x))g'(x)$$

Here the multiplication on the right hand side is composition (or matrix multiplication)

Problem 7: Show that if the Fréchet derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ exists then the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist as well

Problem 8:

(a) Let $f(t) = (\cos(t), \sin(t))$. Show that there does **NOT** exist c in $[0, 2\pi]$ such that

$$f(2\pi) - f(0) = f'(c)(2\pi - 0)$$

So the Mean-Value Theorem is in general **FALSE** in higher dimensions.

(b) That said, show that in general, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable and there is $M \geq 0$ such that $\|f'(x)\| \leq M$ for all x then for all a and b we have

$$|f(b) - f(a)| \leq M |b - a|$$

In particular, if $f'(x) = 0$ for all x then f is constant.

Problem 9: Use the definition of a derivative in $\mathbb{R}^n$ to give a new proof of the product rule. That is, if $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at x, then so if fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Problem 10: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at every point in $\mathbb{R}^2$ but f is not continuous at $(0, 0)$

Problem 11: Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2}$$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are bounded functions in $\mathbb{R}^2$ but f is not differentiable at $(0, 0)$