HOMEWORK 7 – SOLUTIONS

Problem 1:

$$F(x, y, z, w) = \begin{bmatrix} 3x^2 z + 6wy^2 - 2z + 1 \\ xz - \frac{4y}{z} - 3w - z \end{bmatrix}$$

To use the Implicit Function Theorem, check $\det F_{x,y}(1, 2, -1, 0) \neq 0$ (the derivative with respect to what you want to solve for is nonzero)

$$F_{x,y} = \begin{bmatrix} 6xz & 12wy \\ z & -\frac{4}{z} \end{bmatrix}$$

$$F_{x,y}(1, 2, -1, 0) = \begin{bmatrix} 6(1)(-1) & 12(0)(2) \\ -1 & -\frac{4}{-1} \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\det F_{x,y}(1, 2, -1, 0) = -6(4) - 0 = -24 \neq 0$$

Therefore the Implicit Function Theorem says that there is G such that $(x, y) = G(z, w)$ near $(1, 2, -1, 0)$. Moreover

$$G'(-1, 0) = -\left(F_{x,y}(1, 2, -1, 0)\right)^{-1} \left(F_{z,w}(1, 2, -1, 0)\right)$$

$$F_{z,w} = \begin{bmatrix} 3x^2 - 2 & 6y^2 \\ x + \frac{4y}{z^2} - 1 & -3 \end{bmatrix}$$

$$F_{z,w}(1, 2, -1, 0) = \begin{bmatrix} 3(1)^2 - 2 & 6(2)^2 \\ 1 + \frac{4(2)}{(-1)^2} - 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 24 \\ 8 & -3 \end{bmatrix}$$

$$G'(-1, 0) = -\begin{bmatrix} -6 & 0 \\ -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 24 \\ 8 & -3 \end{bmatrix} = -\left(-\frac{1}{24}\right) \begin{bmatrix} 4 & 0 \\ 1 & -6 \end{bmatrix} \begin{bmatrix} 1 & 24 \\ 8 & -3 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 4 & 96 \\ -47 & 42 \end{bmatrix}$$

Problem 2:

Consider $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ given by $F(x, y) = y - f(x)$, where $x, y \in \mathbb{R}^n$.

Let $a \in \mathbb{R}^n$ with $\det(f'(a)) \neq 0$, and let $(x_0, y_0) = (a, f(a))$. Note

$$F(x_0, y_0) = F(a, f(a)) = f(a) - f(a) = 0.$$

Also,

$$F_x(x_0, y_0) = \lim_{t \to 0} \frac{F(x_0 + t, y_0) - F(x_0, y_0)}{t} = \lim_{t \to 0} \frac{f(a) - f(a + t)}{t},$$

so $F_x(x_0, y_0) = -f'(a)$ and thus $\det(F_x(x_0, y_0)) \neq 0$. Therefore, by the implicit function theorem, there exists an open neighborhood W of $(a, f(a))$, an open neighborhood V of $f(a)$, and a function $g : V \to \mathbb{R}^n$ such that

$$\{(x, y) \in W | F(x, y) = 0\} = \{(g(y), y) | y \in V\}.$$

That is, for $y \in V$, we have $F(g(y), y) = y - f(g(y)) = 0$, so $f \circ g$ is the identity on V. We may take the range U to be $f^{-1}(V)$, which is open (since f is continuous) and contains a.

It remains to show $g'(f(a)) = (f'(a))^{-1}$. The implicit function theorem tells us

$$g'(f(a)) = g'(y_0) = -(F_x(x_0, y_0))^{-1} F_y(x_0, y_0) = (f'(a))^{-1},$$

since $F_y(x_0, y_0) = 1$.

Problem 3:

Let $f(t) = t + 2t^2 \sin(1/t)$ with $f(0) = 0$.

Claim: $f'(0) = 1$.

Since $f(0) = 0$, we have

$$f'(0) = \lim_{t \to 0} \frac{f(t)}{t} = \lim_{t \to 0} (1 + 2t \sin(1/t)) = 1.$$

Claim: f' is bounded in $(-1, 1)$.

We can compute directly

$$f'(t) = 1 + 4t \sin\left(\frac{1}{t}\right) - 2\cos\left(\frac{1}{t}\right)$$

for $t \neq 0$. Thus $|f'(t)| \leq 7$ for $t \in (-1, 1)$.

Claim: f is not one-to-one in any neighborhood of 0.

Given a neighborhood U of 0, choose an integer m large enough that $\left[\frac{1}{2m\pi}, \frac{1}{(2m-1)\pi}\right] \subset U$. Note Note that for $k \in \mathbb{Z}$,

$$f'\left(\frac{1}{k\pi}\right) = 1 + \frac{4}{k\pi}\sin(k\pi) - 2\cos(k\pi) = \begin{cases} -1 & \text{if } k \text{ is even} \\ 3 & \text{if } k \text{ is odd} \end{cases}.$$

Therefore, f is decreasing at $\frac{1}{2m\pi}$ and increasing at $\frac{1}{(2m-1)\pi}$, and thus (since f is continuous on I_m) attains its minimum somewhere on the interior of I_m. In particular, f is not one-to-one on I_m.

Problem 4:

Let $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = e^x \sin y$, and let $f = (f_1, f_2)$.

Claim: The Jacobian of f is not zero at any point of $\mathbb{R}^2$; nevertheless, f is not one-to-one on $\mathbb{R}^2$.

The Jacobian of f at (x,y) is

$$\det Df(x,y) = \det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x}\cos^2 y + e^{2x}\sin^2 y = e^{2x},$$

which is nonzero everywhere.

On the other hand, $f(0,0) = f(0,2\pi)$, so f is not one-to-one.

Claim: Let $a = (0, \frac{\pi}{3})$ and $b = f(a)$. Then

$$g(u,v) = \left(\ln\sqrt{u^2+v^2}, \arctan\frac{v}{u}\right)$$

is continuous and satisfies $f(g(y)) = y$ for all y in a neighborhood of b.

Lt us derive g. Let $u = e^x \cos y$, $v = e^x \sin y$. From $u^2 + v^2 = e^{2x}$ we obtain $x = \ln\sqrt{u^2+v^2}$. From $v/u = \tan y$ we obtain $y = \arctan(v/u)$ (for y near $\frac{\pi}{3}$). Then g as defined above is continuous and satisfies $f(g(u,v)) = (u,v)$, and $g(b) = a$.

Additionally, let us compute $f'(a)$ and $g'(b)$ directly and verify directly that $f'(g(u,v))g'(u,v) = I$. We have

$$f'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}, \quad g'(u,v) = \begin{pmatrix} \frac{u}{u^2+v^2} & \frac{v}{u^2+v^2} \\ \frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{pmatrix}.$$

Therefore (since $b = f(a) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$)

$$f'(a) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad g'(b) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Also,

$$f'(g(u,v)) = f'\left(\ln\sqrt{u^2+v^2}, \arctan\frac{u}{v}\right) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}.$$

Multiplying matrices, $f'(g(u,v))g'(u,v) = I$ is verified.

Problem 5:

Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 and consider $F : X \to X$ where $X = C([a,b])$ with its sup norm, defined by

$$F(u)(t) = f(u(t)).$$

We will show F is differentiable and in fact the derivative is given by $DF_u(h)(t) = L_u(h)(t) = f'(u(t))h(t)$.

First we show L_u (as defined above) is a bounded linear operator.

It is clear that L_u is linear in its argument h. For boundedness, let $M = \sup_{t\in[a,b]} |f'(u(t))|$ (which exists because $f' \circ u$ is continuous and $[a,b]$ is compact), and note

$$\|L_u(h)\| = \sup_{t\in[a,b]} |f'(u(t))h(t)| \le M\|h\|.$$

Thus L_u is bounded.

Now we show that L_u is in fact the derivative DF_u at u.

For fixed u, note $u([a,b])$ is compact, hence contained in some interval $[c,d]$. Then the compact interval $K = [c-1, d+1]$ contains $u(t)$ and $u(t)+h$ for all t and all h with $|h| < 1$. Note f' is uniformly continuous on K.

Fix $\epsilon > 0$. Notice that

$$\|F(u+h)-F(u)-L_u(h)\| = \sup_{t\in[a,b]} |f(u(t)+h(t))-f(u(t))-f'(u(t))h(t)|.$$

Since f' is uniformly continuous on K, we may choose $\delta < 1$ so $|f'(x)-f'(y)| < \epsilon$ when $|x-y| < \delta$ and $x, y \in K$. Now by the mean value theorem, for any x and h, we have

$$f(x+h) - f(x) = f'(z)h$$

for some z between x and $x+h$. By the uniform continuity of f' on K, for $x \in u([a,b])$,

$$|f(x+h) - f(x) - f'(x)h| = |f'(z) - f'(x)||h| < \epsilon|h|$$

whenever $|h| < \delta$.

Now setting $x = u(t)$ and $h = h(t)$, we find

$$|f(u(t)+h(t)) - f(u(t)) - f'(u(t))h(t)| < \epsilon\|h\|$$

whenever $\|h\| < \delta$. Thus

$$\|F(u+h) - F(u) - L_u(h)\| < \epsilon\|h\|$$

whenever $\|h\| < \delta$. It follows

$$\lim_{h\to 0} \frac{\|F(u+h) - F(u) - L_u(h)\|}{\|h\|} = 0,$$

so L_u is the derivative of F at u.

Problem 6:

Claim: X_ν is a Banach space with respect to $\|\cdot\|$.

First we need to establish that $\|\cdot\|$ is actually a norm. It is clear that $\|cu\| = |c|\|u\|$ and that $\|u\| \geq 0$ with equality if and only if $u = 0$. For the triangle inequality,

$$\|u+v\| = \sup_{t\geq 0} \|(u(t)+v(t))e^{\nu t}| \leq \sup_{t\geq 0} |u(t)e^{\nu t}| + \sup_{t\geq 0} |v(t)e^{\nu t}| = \|u\|+\|v\|.$$

Now we need to show completeness. Suppose $(u_n) \subset X_\nu$ is Cauchy. Then (u_n) has at least a pointwise limit u (since $(u_n(t))$ is Cauchy in $\mathbb{R}$); we must show u is in fact the limit of (u_n) in X_ν and that $u \in X_\nu$. To this end, fix $\epsilon > 0$, and choose N large enough that for $m, n > N$, we have $\|u_n - u_m\| < \epsilon/2$. Now for any *fixed* t, there exists $m > N$ (depending on t) such that $|u(t) - u_m(t)|e^{\nu t} < \epsilon/2$, so for any $n > N$ (where N does not depend on t) we have

$$|u(t)e^{\nu t} - u_n(t)e^{\nu t}| \leq |u(t)e^{\nu t} - u_m(t)e^{\nu t}| + |u_m(t)e^{\nu t} - u_n(t)e^{\nu t}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\|u - u_n\| \to 0$, which also implies (for large enough n)

$$\|u\| \leq \|u - u_n\| + \|u_n\| < \infty.$$

All that remains is to show that u is continuous. Again, fix $\epsilon > 0$. Fix $t \geq 0$; we must find δ so that if $|t - s| < \delta$ then $|u(t) - u(s)| < \epsilon$. Choose n large enough that $\|u - u_n\| < \frac{\epsilon}{6} e^{\nu t}$. Then choose δ small enough that $|u_n(s) - u_n(t)| < \frac{\epsilon}{2}$, and also that $e^{|\nu|\delta} < 2$. Then

$$\begin{aligned}
|u(s) - u(t)| &\leq |u(s) - u_n(s)| + |u_n(s) - u_n(t)| + |u_n(t) - u(t)| \\
&< \frac{\epsilon}{2} + (e^{-\nu s} + e^{-\nu t})\|u - u_n\| \\
&\leq \frac{\epsilon}{2} + (1 + e^{|\nu|\delta})e^{-\nu t}\|u - u_n\| \\
&\leq \frac{\epsilon}{2} + 3e^{-\nu t}\|u - u_n\| \\
&< \epsilon.
\end{aligned}$$

This establishes that u is continuous, so together with $\|u\| < \infty$, we have $u \in X_\nu$.

Claim: Assume $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g(0) = 0$ and is Lipschitz continuous with Lipschitz constant L. Then $G : X_\nu \to X_\nu$ defined by

$$[G(u)](t) = g(u(t))$$

is well-defined and Lipschitz continuous with Lipschitz constant L.

It is clear $G(u)$ is a well-defined continuous function, so we must show $G(u) \in X_\nu$. We have for any $u \in X_\nu$ that

$$\|G(u)\| = \sup_{t\geq 0} e^{\nu t}|g(u(t))| \leq \sup_{t\geq 0} e^{\nu t} L|u(t)| = L\|u\| < \infty$$

This establishes $G(u) \in X_\nu$. Furthermore, for $u, v \in X_\nu$,

$$\begin{aligned}\|G(u) - G(v)\| &= \sup_{t\geq 0} e^{\nu t}|g(u(t)) - g(v(t))| \\ &\leq \sup_{t\geq 0} e^{\nu t} L|u(t) - v(t)| \\ &= L\|u - v\|.\end{aligned}$$

Thus G is Lipschitz continuous with Lipschitz constant L. This is the best Lipschitz constant we can get in general: if $g(x) = Lx$, then g has best Lipschitz constant L and all the inequalities above are equalities, so G has best Lipschitz constant L as well.

Claim: For $\nu \geq 0$, $g \in C^1$ implies $G \in C^1$. In fact, G has derivative $DG_u(h)(t) = K_u(h)(t) = g'(u(t))h(t)$, much like in the previous problem. The argument proceeds very similarly.

First note that every $u \in X_\nu$ is bounded because $\nu \geq 0$. Thus for each fixed $u \in X_\nu$, there exists C so $|g'(u(t))| \leq C$ for all $t \geq 0$. Therefore

$$\|K_u(h)\| = \sup_{t\geq 0} |g'(u(t))h(t)|e^{\nu t} \leq C\|h\|,$$

establishing that K_u is a bounded linear functional.

Next, since $\nu \leq 0$, $u \in X_\nu$ implies u is bounded, that is there exists M so $|u(t)| \leq M$ for all $t \geq 0$. Then the compact set $[-M-1, M+1]^n$ contains $u(t)$ and $u(t)+h$ for all $t \geq 0$ and h with $|h| < 1$. Since g' is continuous, it is uniformly continuous on $[-M-1, M+1]^n$.

Fix $\epsilon > 0$. By the argument in the previous problem, the uniform continuity of g' on $[-M-1, M+1]^n$ implies there is δ so that $|g(x+h) - g(x) - g'(x)h| < \epsilon|h|$ whenever $|h| < \delta$ and $|x| \leq M$. Setting $x = u(t)$ and $h = h(t)$, we find that for $\|h\| < \delta$ (which implies $|h(t)| < \delta$ for all $t \geq 0$, since $\nu \geq 0$) that

$$\begin{aligned}
&\|G(u+h) - G(u) - K_u(h)\| \\
&\qquad = \sup_{t\geq 0} e^{\nu t}|g(u(t)+h(t)) - g(u(t)) - g'(u(t))h(t)| \\
&\qquad \leq \sup_{t\geq 0} e^{\nu t}\epsilon|h(t)| \\
&\qquad = \epsilon\|h\|.
\end{aligned}$$

We can now conclude that K_u is the derivative DG_u of G at u.

All that remains is to show that DG is a continuous function of $u \in X_\nu$. Fix $u \in X_\nu$ and let $M = \sup_{t\geq 0}|u(t)|$, which is finite. Note that

$$\|DG_u - DG_v\| = \sup_{\|h\|\leq 1}\sup_{t\geq 0} |(g'(u(t)) - g'(v(t)))h(t)|e^{\nu t}.$$

Fix $\epsilon > 0$. By uniform continuity of g' on $[-M-1, M+1]^n$, choose $\delta < 1$ such that $|g'(x) - g'(y)| < \epsilon$ whenever $|x-y| < \delta$ and $|x|, |y| \leq M+1$.

If $\|u-v\|<\delta$, then $|u(t)-v(t)|<\delta$ for all t (again since $\nu \geq 0$, so $|u(t)|, |v(t)| \leq M+1$ for all t as well. Thus

$$\|DG_u - DG_v\| \leq \sup_{\|h\|\leq 1} \sup_{t\geq 0} \epsilon |h(t)| e^{\nu t} = \epsilon \sup_{\|h\|\leq 1} \|h\| = \epsilon.$$

We conclude DG is a continuous function of u, so G is C^1.