ANALYSIS – HOMEWORK 7

Problem 1: Consider the system

$$\begin{cases} 3x^2z + 6wy^2 - 2z + 1 = 0 \\ xz - \dfrac{4y}{z} - 3w - z = 0 \end{cases}$$

Show that you can solve for x and y in terms of z and w around the point $(1, 2, -1, 0)$ and calculate $G'(-1, 0)$ (where G is the graph of x, y in terms of z, w)

Problem 2: Show that the Implicit Function Theorem implies (the following version of) the Inverse Function Theorem:

Inverse Function Theorem:

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. If $\det(f'(a)) \neq 0$ for some a, there is an open neighborhood U of a and an open neighborhood V of $f(a)$ and $g : V \to U$ such that $f(g(y)) = y$ for all y. Moreover

$$g'(f(a)) = (f'(a))^{-1}$$

Note: Hence, combined with the proof given in class, the Inverse and Implicit function theorems are equivalent.

Problem 3: Show that if $f(0) = 0$ and

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

Then $f'(0) = 1$, f' is bounded in $(-1, 1)$ but f is not one-to-one in any neighborhood of 0.

In other words, the continuity of f' at a is really needed in the inverse function theorem!

Problem 4: Let $f = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by

$$\begin{aligned} f_1(x,y) =& e^x \cos(y) \\ f_2(x,y) =& e^x \sin(y) \end{aligned}$$

(a) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus every point of $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, show that f is not one-to-one on $\mathbb{R}^2$

(b) Let $a = \left(0, \frac{\pi}{3}\right)$ and $b = f(a)$. Let g be the continuous inverse of f, defined in a neighborhood of b such that $g(b) = a$. Find an explicit formula for g, compute $f'(a)$ and $g'(b)$ and verify that $f'(g(x,y))g'(x,y) = I$ holds in this case

Problem 5: Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 and consider $F : X \to X$ where $X = C([a,b])$ with its sup norm, defined by

$$F(u)(t) = f(u(t))$$

Show that F is differentiable and find DF

(Turn page)

Problem 6: For each fixed $\nu \in \mathbb{R}$ consider the space

$$X_\nu =: \left\{ u : [0, \infty) \to \mathbb{R}^n \mid u \text{ is continuous and } \|u\| =: \sup_{t \geq 0} |u(t)| \, e^{\nu t} < \infty \right\}$$

Notice in particular that $|u(t)| \leq \|u\| \, e^{-\nu t}$ for all $t \geq 0$

(a) Prove that X_ν is a Banach space with respect to the norm $\|\cdot\|$

(b) Assume that $g : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g(0) = 0$ and is Lipschitz continuous. Show that $G : X_\nu \to X_\nu$ defined by

$$[G(u)](t) = g(u(t))$$

Is well-defined and Lipschitz continuous. How are the Lipschitz constants of g and G related?

(c) For $\nu \geq 0$, prove that $g \in C^1$ implies $G \in C^1$