HOMEWORK 8 – SOLUTIONS

Note: Here the x_k and t_k are switched (compared to the notes)

Problem 1:

STEP 1: Partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$$

STEP 2: $U(f, P)$

Since x^2 is increasing, notice that:

$$\sup_{t \in [t_{k-1}, t_k]} f(t) = f(t_k) = (t_k)^2 \quad \text{(Right Endpoint)}$$

$$U(f, P) = \sum_{k=1}^{n} (t_k)^2 (t_k - t_{k-1})$$

STEP 3: $U(f)$

Given n, let P be the evenly spaced Calculus partition with $t_k = \frac{k}{n}$:

In that case $t_k - t_{k-1} = \frac{1}{n}$ and

$$\begin{aligned} U(f,P) =& \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{1}{n}\right) \\ =& \sum_{k=1}^{n} \frac{k^2}{n^3} \\ =& \frac{1}{n^3} \sum_{k=1}^{n} k^2 \\ =& \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\ =& \frac{(n+1)(2n+1)}{6n^2} \end{aligned}$$

Upshot: Since $U(f)$ is the inf over all partitions, we must have

$$U(f) \leq U(f,P) = \frac{(n+1)(2n+1)}{6n^2}$$

Therefore, taking the limit as $n \to \infty$ of the right hand side[1], we get $U(f) \leq \frac{2}{6} = \frac{1}{3}$, and so $U(f) \leq \frac{1}{3}$

STEP 4: $L(f)$

This is similar to the above, except that here $\inf_{t \in [t_{k-1}, t_k]} f(t) = (t_{k-1})^2$ (Left endpoint), and so, using sup we get $L(f) \geq \frac{1}{3}$.

Since $U(f) \leq \frac{1}{3} \leq L(f)$ and because $L(f) \leq U(f)$, we get $L(f) = U(f) = \frac{1}{3}$. Hence $f(x) = x^2$ is Darboux integrable and $\int_0^1 x^2 dx = \frac{1}{3}$.

Problem 2:

[1]Here we used that if $a \leq s_n$, then so is $a \leq s$, where s is the limit of s_n

Proof: WLOG, assume f is strictly increasing, and so $f(a) < f(b)$

Main Observation: In that case, we have

$$\sup_{t\in[t_{k-1},t_k]} f(t) = f(t_{k-1}) \text{ and } \inf_{t\in[t_{k-1},t_k]} f(t) = f(t_k)$$

In order to show f is integrable, let's use the Darboux integrability criterion

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{f(b)-f(a)}$ and and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be any partition with mesh $< \delta$, then:

$$\begin{aligned}
U(f,P) - L(f,P) =& \sum_{k=1}^{n} f(t_k)(t_k - t_{k-1}) - \sum_{k=1}^{n} f(t_{k-1})(t_k - t_{k-1}) \\
=& \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right)(t_k - t_{k-1}) \\
<& \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right) \frac{\epsilon}{f(b) - f(a)} \\
=& \frac{\epsilon}{f(b) - f(a)} \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) \\
=& \left(\frac{\epsilon}{f(b) - f(a)}\right)(f(t_n) - f(t_0)) \qquad \text{(Telescoping sum)} \\
=& \left(\frac{\epsilon}{f(b) - f(a)}\right)(f(b) - f(a)) \\
=& \epsilon \checkmark
\end{aligned}$$

Hence f is integrable $\square$

Problem 3:

Proof:

($\Rightarrow$) Let $\epsilon > 0$ be given, then and consider:

$$L(f) - \frac{\epsilon}{2} < L(f) = \sup \{ L(f, P) \mid P \text{ partition } \}$$

By def of sup, there is a partition P_1 such that $L(f, P_1) > L(f) - \frac{\epsilon}{2}$

Similarly there is a partition P_2 such that $U(f, P_2) < U(f) + \frac{\epsilon}{2}$

Let $P = P_1 \cup P_2$ (finer), then $L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$, and therefore:

$$\begin{aligned} U(f, P) - L(f, P) \leq & U(f, P_2) - L(f, P_1) \\ < & U(f) + \frac{\epsilon}{2} - \left(L(f) - \frac{\epsilon}{2} \right) \\ = & \underbrace{U(f) - L(f)}_{0} + \epsilon \\ = & \epsilon \end{aligned}$$

Here we used $U(f) = L(f)$, since f is integrable ✓

($\Leftarrow$) Let $\epsilon > 0$ be given and let P be such that $U(f, P) - L(f, P) < \epsilon$. Then by definition of $U(f)$ as an inf, we get:

$$\begin{aligned} U(f) \leq & U(f, P) \\ = & U(f, P) - L(f, P) + L(f, P) \\ < & \epsilon + L(f, P) \\ \leq & \epsilon + L(f) \end{aligned}$$

Hence $U(f) < L(f) + \epsilon$ for all $\epsilon > 0$, hence $U(f) \leq L(f)$, but since $L(f) \leq U(f)$ as well, we get $U(f) = L(f)$ ✓ □

Problem 4:

Beautiful application of uniform continuity!

Since f is continuous on $[a, b]$, it is uniformly continuous on $[a, b]$

Let $\epsilon > 0$ be given, then there is $\delta > 0$ such that for all x and y, if $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$

Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be any part. with $\text{mesh}(P) < \delta$.

Since f is continuous on each sub-piece $[t_{k-1}, t_k]$, it attains a maximum and a minimum for some x_k and y_k in $[t_{k-1}, t_k]$

Therefore, by definition,

$$\sup_{t \in [t_{k-1}, t_k]} = f(x_k) \text{ and } \inf_{t \in [t_{k-1}, t_k]} = f(y_k)$$

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{k=1}^{n} \left(f(x_k) - f(y_k)\right)(t_k - t_{k-1}) \\
&\leq \sum_{k=1}^{n} |f(x_k) - f(y_k)| (t_k - t_{k-1}) \\
&< \sum_{k=1}^{n} \left(\frac{\epsilon}{b-a}\right)(t_k - t_{k-1}) \qquad \text{(Uniform Continuity)} \\
&= \frac{\epsilon}{b-a} \sum_{k=1}^{n} t_k - t_{k-1} = \left(\frac{\epsilon}{b-a}\right)(b-a) = \epsilon ✓
\end{aligned}$$

Hence, by the Darboux Integrability Criterion, f is integrable on $[a, b]$

□

Problem 5:

Fix a partition P, then for any x and y in a given sub-piece $[t_{k-1}, t_k]$, we have

$$\begin{aligned}
(f(x))^2 - (f(y))^2 =& (f(x) + f(y))(f(x) - f(y)) \\
\leq& |f(x) + f(y)| |f(x) - f(y)| \\
\leq& (|f(x)| + |f(y)|) |f(x) - f(y)| \\
\leq& (B + B) |f(x) - f(y)| \\
=& 2B |f(x) - f(y)|
\end{aligned}$$

Here $B = \sup_x |f(x)|$

Then, taking the sup over $x \in [t_{k-1}, t_k]$ and then the inf over $y \in [t_{k-1}, t_k]$, we get

$$\begin{aligned}
\sup_{t \in [t_{k-1}, t_k]} f^2 - \inf_{t \in [t_{k-1}, t_k]} f^2 \leq& 2B \left| \sup_{t \in [t_{k-1}, t_k]} f - \inf_{t \in [t_{k-1}, t_k]} f \right| \\
=& 2B \left(\sup_{t \in [t_{k-1}, t_k]} f - \inf_{t \in [t_{k-1}, t_k]} f \right)
\end{aligned}$$

Finally, summing over k, we get

$$U(f^2, P) - L(f^2, P) \leq 2B \left(U(f, P) - L(f, P) \right)$$

Now, let $\epsilon > 0$ be given, then since f is integrable on $[a, b]$, by the Darboux Integrability Criterion, there is a partition P such that

$U(f,P) - L(f,P) < \frac{\epsilon}{2B}$.

With the same P, we get

$$U(f^2,P) - L(f^2,P) \leq 2B\left(U(f,P) - L(f,P)\right) < (2B)\left(\frac{\epsilon}{2B}\right) = \epsilon\checkmark$$

Hence, by the Darboux Integrability criterion again, f^2 is integrable on $[a,b]$

For the counterexample, let $f(x) = \frac{1}{\sqrt{x}}$ then

$$\int_0^1 f(x)dx = \int_0^1 \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_0^1 = 2$$

But $f^2 = \frac{1}{x}$ and

$$\int_0^1 f^2(x)dx = \int_0^1 \frac{1}{x} = \left[\ln|x|\right]_0^1 = \infty$$

Problem 6:

Proof: The idea is to choose a clever x_k that makes the Riemann sum equal to $f(b) - f(a)$

Let P be any partition of $[t_{k-1}, t_k]$

By the MVT, for every k, there is x_k in $[t_{k-1}, t_k]$ such that

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}$$

This implies

$$f'(x_k)\,(t_k - t_{k-1}) = f(t_k) - f(t_{k-1})$$

With *this* choice of x_k the Riemann sum of f' becomes

$$\begin{aligned}\sum_{k=1}^{n} f'(x_k)\,(t_k - t_{k-1}) =& \sum_{k=1}^{n} f(t_k) - f(t_{k-1}) \\ =& f(t_1) - f(t_0) + f(t_2) - f(t_1) + \cdots + f(t_n) - f(t_{n-1}) \\ =& f(t_n) - f(t_0) \\ =& f(b) - f(a)\end{aligned}$$

Since this is true for *any* partition P, we get

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \square$$

Problem 7:

Proof: Beautiful application of uniform continuity (!)

STEP 1: Scratchwork

Our goal is to show that $F'(x) = f(x)$, that is

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} - f(x) \right| = \left| \frac{\int_x^{x+h} f(t)dt}{h} - f(\right.$$

Clever Observation: It would be nice if we could write $f(x)$ as an integral of the same form, but notice that:

$$f(x) = \frac{\int_x^{x+h} f(x)dt}{h}$$

Why? Since $f(x)$ doesn't depend on t, we get

$$\int_x^{x+h} f(x)dt = f(x)\int_x^{x+h} 1\,dt = f(x)(x+h-x) = f(x)h$$

And solving for $f(x)$, we get the desired identity.

Continuing, we get:

$$\begin{aligned}\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| &= \left|\frac{\int_x^{x+h} f(t)dt}{h} - \frac{\int_x^{x+h} f(x)dt}{h}\right| \\ &= \left|\frac{\int_x^{x+h} f(t)-f(x)dt}{h}\right| \\ &\leq \frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt\end{aligned}$$

(WLOG, assume $h > 0$ here)

And **this** is where continuity kicks in!

STEP 2: Actual Proof

Let $\epsilon > 0$ be given

Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$, and so there is $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x)-f(y)| < \epsilon$.

With the same δ, if $0 < h < \delta$ then $|f(t) - f(x)| < \epsilon$

Why? If t is in $[x, x+h]$ then $|x - t| \leq h < \delta$, and so $|f(t) - f(x)| < \epsilon$

We can continue the calculation to get

$$\begin{aligned}
\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq & \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\, dt \\
< & \frac{1}{h}\int_x^{x+h} \epsilon \, dt \\
= & \frac{\epsilon}{h}(x + h - x) \\
= & \epsilon\left(\frac{h}{h}\right) \\
= & \epsilon
\end{aligned}$$

Hence if $0 < h < \delta$, then $\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| < \epsilon$

Therefore $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$ that is $F'(x) = f(x)$ □

(Technically we've only shown the limit as $h \to 0^+$, but the other limit is similar)

Problem 8:

$$\begin{aligned}\int_0^1 x\tan^{-1}(x)dx &= \left[\left(\frac{x^2+1}{2}\right)\tan^{-1}(x)\right]_0^1 - \int_0^1 \left(\frac{x^2+1}{2}\right)\left(\frac{1}{x^2+1}\right)dx \\ &= \left(\frac{2}{2}\right)\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(0) - \int_0^1 \frac{1}{2}dx \\ &= \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$