# HOMEWORK 9 – SOLUTIONS

## Problem 1:

**STEP 1:** Define the following equivalence relation on $[0,1]$:

$$x \sim y \Leftrightarrow x - y \text{ is rational}$$

Using $\sim$ we can partition $[0,1]$ into equivalence classes, that is we can write $[0,1]$ as a **disjoint** union

$$[0,1] = \bigcup_{a \in [0,1]} [a]$$

Where $[a] = \{x \mid x \sim a\}$

**STEP 2:** For every equivalence class $[a]$, choose **exactly** one element $x_a$ from each equivalence class, and let

$$\mathcal{N} = \{x_a\}$$

(This "choosing" step requires the axiom of choice)

**STEP 3:** $\mathcal{N}$ is not measurable.

By contradiction, suppose $\mathcal{N}$ is measurable.

Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of all the rationals in $[-1,1]$ and consider the translates

$$\mathcal{N}_k =: \mathcal{N} + r_k$$

We claim that $\mathcal{N}_k$ are disjoint and

$$[0,1] \subseteq \bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1,2]$$

**Disjoint:** Suppose $\mathcal{N}_k \cap \mathcal{N}_p \neq \emptyset$. Then there are rationals $r_k \neq r_p$ and $a$ and $b$ such that $x_a + r_k = x_b + r_p$ but then $x_a - x_b = r_p - r_k \in \mathbb{Q}$ and hence $x_a \sim x_b$ which contradicts the fact that we chose *exactly* one element from each equivalence class

**Inclusions:** If $x \in [0,1]$ then $x \sim x_a$ for some $a$ and hence $x - x_a = r_k$ for some $k$ and so $x \in \mathcal{N}_k$ and the second inclusion holds since each $\mathcal{N}_k$ is contained in $[-1,2]$ by construction

**STEP 4: Conclusion**

If each $\mathcal{N}$ were measurable, then so would $\mathcal{N}_k$ for all $k$ (by translation) and since the union $\bigcup_{k=1}^{\infty} \mathcal{N}_k$ is disjoint, the above would imply:

$$m([0,1]) \leq m\left(\bigcup_{k=1}^{\infty} \mathcal{N}_k\right) \leq \quad m([-1,2])$$
$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq 3$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we have $m(\mathcal{N}_k) = m(\mathcal{N})$ and hence

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3$$

Hence a contradiction, since neither $m(\mathcal{N}) = 0$ or $m(\mathcal{N}) > 0$ holds $\square$

**Problem 2:**

**Part (a):** Since $E \subset F$, we can write $F$ as the disjoint union of $E$ and $F \setminus E$. Therefore by additivity of measure

$$\mu(E) + \mu(F \setminus E) = \mu(F)$$

By nonnegativity of measure, $\mu(F \setminus E) \geq 0$, so $\mu(E) \leq \mu(F)$. Also, if $\mu(E) < \infty$, we can subtract it from both sides to get $\mu(F \setminus E) = \mu(F) - \mu(E)$.

**Part (b):** For each $n$, let

$$F_n = E_n \setminus \bigcup_{m<n} E_m.$$

Then the $F_n$ are disjoint and measurable and their union is the same as the union of the $E_n$. Also, $F_n \subset E_n$, so $\mu(F_n) \leq \mu(E_n)$. Therefore, using countable additivity of measure,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

**Part (c):** For each $n \geq 2$, let $F_n = E_n \setminus E_{n-1}$, and let $F_1 = E_1$. Then the $F_n$ are disjoint and measurable and their union is the same as the union of the $E_n$. Also, each $E_n$ is the disjoint union of the $F_m$ for $m \leq n$, so

$$\mu(E_n) = \sum_{m=1}^{n} \mu(F_n);$$

furthermore, the union of all the $E_n$ is the same as the union of all the $F_n$. Therefore, applying countable additivity of measure,

$$\lim_{n\to\infty} \mu(E_n) = \sum_{n=1}^{\infty} \mu(F_n) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right).$$

**Part (d):** Let $F_n = E_1 \setminus E_n$, so the $F_n$ are an increasing sequence of sets and thus the previous part implies

$$\lim_{n\to\infty} \mu(F_n) = \mu\left(\bigcup_{n=1}^{\infty} F_n\right).$$

Since $\mu(E_n) \leq \mu(E_1) < \infty$, we have $\mu(F_n) = \mu(E_1) - \mu(E_n)$ by part (a). Now consider the right hand side: the union of all the $F_n$ is $E_1$ minus the intersection of all the $E_n$, which is contained in $E_1$ and thus has finite measure. Applying part (a) again,

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Therefore,

$$\lim_{n\to\infty} (\mu(E_1) - \mu(E_n)) = \mu(E_1) - \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

Since $\mu(E_1) < \infty$, we can subtract it from both sides and conclude

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

**Problem 3:**

**Part (a):** We can write $E$ as

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

This will imply $E$ is measurable.

For the inclusion in the direction $\subset$, note $x \in E$ means there are infinitely many $k$ for which $x \in E_k$. Thus for any $n$, there is some $k \geq n$ such that $x \in E_k$. It follows

$$x \in \bigcup_{k=n}^{\infty} E_k$$

for all $n$, establishing the $\subset$ inclusion.

For the other inclusion (in the direction $\supset$), suppose

$$x \notin \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

Then there is some $n$ for which $x \notin E_k$ for any $k \geq n$. Thus $x \in E_k$ for at most $n$ values $k$, so $x \notin E$.

**Part (b):** Define

$$F_n = \bigcup_{k=n}^{\infty} E_k.$$

Then the $F_n$ are non-increasing. Since by problem 2(b)

$$\mu(F_1) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k) < \infty,$$

we can apply problem 2(d) to conclude

$$\lim_{n\to\infty} \mu(F_n) = \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \mu(E).$$

So all that remains is to show $\mu(F_n) \to 0$. Applying problem 2(b) again, we have

$$\mu(F_n) = \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \le \sum_{k=n}^{\infty} \mu(E_k)$$

By assumption, $\sum_{k=1}^{n} \mu(E_k)$ approaches a finite limit $\sum_{k=1}^{\infty} \mu(E_k)$ as $n \to \infty$. Thus

$$\sum_{k=n}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k) - \sum_{k=1}^{n-1} \mu(E_k) \to 0$$

as $n \to \infty$, and we are done.

## Problem 4:

### Part (a):

For $f+g$: Let $\alpha \in \mathbb{R}$. For any real $u, v$ with $u + v > \alpha$, there exists rational $q \in (\alpha - v, u)$. Thus

$$\begin{aligned}(f+g)^{-1}((\alpha,\infty)) &= \{x : f(x) + g(x) > \alpha\} \\ &= \bigcup_{q\in\mathbb{Q}} \{x : f(x) > q > \alpha - g(x)\} \\ &= \bigcup_{q\in\mathbb{Q}} (f^{-1}((q,\infty)) \cap g^{-1}((\alpha - q, \infty))),\end{aligned}$$

which is measurable; hence $f + g$ is measurable.

For $fg$: Since continuous functions are Borel measurable and compositions of measurable functions are measurable, together with part (a) we have that $(f-g)^2$ and $(f+g)^2$ are measurable. Writing

$$fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$$

and applying part (a), we conclude $fg$ is measurable.

**Part (b):**

For $\max\{f,g\}$: Let $\alpha \in \mathbb{R}$. Note $\max\{f(x),g(x)\} > \alpha$ if and only if either $f(x) > \alpha$ or $g(x) > \alpha$. We therefore have

$$(\max\{f,g\})^{-1}((\alpha,\infty)) = f^{-1}((\alpha,\infty)) \cup g^{-1}((\alpha,\infty)),$$

which is measurable; hence $\max\{f,g\}$ is measurable.

For $\min\{f,g\}$: Write $\min\{f,g\} = -\max\{-f,-g\}$, so measurability follows from the above.

**Part (c):**

For $\sup_n f_n$: Let $\alpha \in \mathbb{R}$. Note $\sup_n f_n(x) > \alpha$ if and only if $f_n(x) > \alpha$ for some $n$. Thus

$$(\sup_n f_n)^{-1}((\alpha,\infty)) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty)),$$

which is measurable; hence $\sup_n f_n$ is measurable.

For $\inf_n f_n$: Write $\inf_n f_n = -\sup_n(-f_n)$, so measurability follows from the above.

**Part (d):**

For $\limsup_n f_n$: Note $\limsup_n f_n = \inf_n \sup_{k\geq n} f_k$, so measurability follows from part (b).

For $\liminf_n f_n$: Analogously, note $\liminf_n f_n = \sup_n \inf_{k \geq n} f_k$, so measurability follows from part (b).

**Part (e):**

For $\lim_n f_n$ (if it exists): When $\lim_n f_n$ exists, it is the same as $\limsup_n f_n$ (or $\liminf_n f_n$), which we have already showed is measurable.

**Problem 5:** Let $\mathcal{N}$ be the non-measurable subset from the first problem.

Given $a \in \mathcal{N}$ define

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

Then each $f_a$ is measurable but

$$f =: \sup_{a \in \mathcal{N}} f_a = \chi_{\mathcal{N}}$$

Which is not measurable since $\left\{\frac{1}{2} \leq f \leq 2\right\} = \mathcal{N}$ which is not measurable

**Problem 6:** Here $\{f_n(x)\}$ is Cauchy, meaning for every $k$ there is $N$ such that if $m, n \geq N$ then $|f_m(x) - f_n(x)| < \frac{1}{k}$

Hence the set in question can be written as

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{m,n=N}^{\infty} \left\{ x \text{ such that } |f_m(x) - f_n(x)| < \frac{1}{k} \right\}$$

And therefore we're done because the union/intersection of measurable sets is measurable

## Problem 7:

Let $\mathcal{N}$ be the non-measurable set from the first problem, and let $\mathcal{N}^c = [0,1] \setminus \mathcal{N}$. Then $m_*(\mathcal{N} \cup \mathcal{N}^c) = m_*([0,1]) = 1$, and $m_*(\mathcal{N}) > 0$ (since $\mathcal{N}$ is non-measurable), so all we need to show is that $m_*(\mathcal{N}^c) = 1$.

Assume, for the sake of contradiction, that $m_*(\mathcal{N}^c) < 1$. Then there is $\epsilon > 0$ and a measurable set $U \subset [0,1]$ containing $\mathcal{N}^c$ such that $m_*(U) < 1 - \epsilon$. Note $U^c \subset \mathcal{N}$, and thus (since $U^c$ is measurable) $m_*(U^c) > \epsilon$.

Consider $U^c + r_k$, where $\{r_k\}_{k=1}^{\infty}$ is the enumeration of the rationals on $[-1,1]$ from problem 1. We have

$$\bigcup_{k=1}^{\infty}(U^c + r_k) \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1,2].$$

But the $U^c + r_k$ are measurable and disjoint (since $U^c + r_k \subset \mathcal{N}_k$, and the $\mathcal{N}_k$ are disjoint), and each has measure at least $\epsilon$, so the measure of the set on the left is infinite. This contradicts the fact that $[-1,2]$ has measure 3; we conclude $m_*(\mathcal{N}^c) = 1$.