ANALYSIS – HOMEWORK 9

Problem 1: A non-measurable set

In this problem, we will construct a non-measurable set on $\mathbb{R}$

Define the following equivalence relation on $[0, 1]$

$$x \sim y \Leftrightarrow x - y \text{ is rational}$$

Using $\sim$ we can partition $[0, 1]$ into equivalence classes, that is we can write $[0, 1]$ as a **disjoint** union

$$[0, 1] = \bigcup_{a \in [0,1]} [a]$$

Where $[a] = \{x \mid x \sim a\}$

For each equivalence class $[a]$ choose **exactly** one element x_a from each equivalence class

$$\mathcal{N} = \{x_a\}$$

(This choosing step requires the axiom of choice)

Problem: Show that $\mathcal{N}$ is not measurable

Hint: Consider the translates $\mathcal{N}_k =: \mathcal{N} + r_k$ where r_k is an enumeration of all the rationals in $[-1, 1]$ and show

$$[0, 1] \subseteq \bigcup_{k=1}^{\infty} \mathcal{N}_k \subseteq [-1, 2]$$

Problem 2: Let X be a set, and μ a measure defined on a σ-algebra $\mathcal{M}$. Prove the following facts

(a) If $E \subset F$, $\mu(E) \le \mu(F)$. In addition, if $\mu(E) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.

(b) For any sequence of sets $\{E_n\}$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \mu(E_n).$$

(c) If $\{E_n\}$ is an increasing sequence of nested sets, i.e. $E_n \subset E_{n+1}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

(d) If $\{E_n\}$ is an decreasing sequence of nested sets, i.e. $E_n \supset E_{n+1}$, and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Problem 3: [The Borel-Cantelli Lemma] Suppose $\{E_k\}$ is a countable family of measurable subsets of $\mathbb{R}$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

$$\text{Define } E = \{x \in \mathbb{R} \,|\, x \in E_k \text{ for infinitely many } k\}$$

(a) Show that E is measurable

(b) Show $m(E) = 0$

Problem 4: Let f, g and $\{f_n\}_{n\in\mathbb{N}}$ be measurable functions from $(X, \mathcal{M})$ to $\mathbb{R}$. Show that the following are measurable

(a) $f + g$ and fg

(b) $\max(f, g)$ and $\min(f, g)$

(c) $\sup_n f_n(x)$ and $\inf_n f_n(x)$

(d) $\limsup_{n\to\infty} f_n(x)$ and $\liminf_{n\to\infty} f_n(x)$.

(e) $\lim_{n\to\infty} f_n(x)$, provided the limit exists.

Problem 5: Show that the supremum of an uncountable family of measurable functions might not be measurable.

Problem 6: If $\{f_n\}$ is a sequence of measurable functions, show that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Problem 7: Find two subsets E_1 and E_2 of $\mathbb{R}$ such that

$$m_\star(E_1 \cup E_2) \neq m_\star(E_1) + m_\star(E_2)$$

Hint: Let $E_1 = \mathcal{N}$ as in the first problem and $E_2 = \mathcal{N}^c = [0, 1]\backslash\mathcal{N}$

Show by contradiction that $m_\star(\mathcal{N}^c) = 1$. You're allowed to assume that in this case for all $\epsilon > 0$ there is $U \subseteq [0, 1]$ measurable such that $\mathcal{N}^c \subseteq U$ and $m_\star(U) < 1-\epsilon$. The translates $\mathcal{N}_k$ (as in the first problem) prove useful here.