LECTURE: COMPACTNESS AND UNIFORM CONVERGENCE

1. Compactness (continued)

Compact Equivalence Theorem:

Let (X, d) be a metric space with $K \subset X$. Then the following are equivalent:

(1) K is covering compact.

(2) K is sequentially compact.

(3) K is complete and totally bounded.

Proof:

(1) $\Longrightarrow$ (2): Last time

(2) $\Longrightarrow$ (3): Suppose that K is sequentially compact.

First, we show that K is complete. Let $\{x_n\}$ be a Cauchy sequence in K. By sequential compactness, $\{x_n\}$ has a subsequence which converges to $x^* \in K$.

Then $x_n \to x^*$ because

$$d(x_n, x^*) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x^*) \to 0$$

As $n, k \to \infty$ by Cauchiness and convergence, and so (x_n) converges ✓

Now suppose K is not totally bounded. Then there exists $\epsilon > 0$ such that K cannot be covered by finitely many open balls $B(x, \epsilon)$. Define a sequence $\{x_n\}$ as follows. Start by choosing any $x_1 \in K$. Then choose

$$\begin{aligned} &x_2 \in K \setminus B(x_1, \epsilon) \\ &x_3 \in K \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon)) \\ &x_3 \in K \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon) \cup B(x_3, \epsilon)) \\ &\quad\vdots \end{aligned}$$

In other words, each element x_k in the sequence lies outside all of the previous ϵ-balls. This process never terminates, otherwise K could in fact be covered by finitely many ϵ-balls. By sequential compactness, $\{x_n\}$ has a convergent subsequence, but this is impossible since $d(x_j, x_k) \geq \epsilon$ for all $j \neq k$.

(3) $\Longrightarrow$ (1): (will be skipped in class)

Suppose K is complete and totally bounded, but not covering compact.

Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of K, and assume that there is no finite subcover. We construct the following sequence of sets.

STEP 1: Take $\epsilon_1 = 1/2$. Since K is totally bounded, we can find points $y_1^1, \ldots, y_{n(1)}^1$ such that

$$K \subset \bigcup_{k=1}^{n(1)} B_{1/2}(y_k^1).$$

Since no finite subcover of $\{U_\alpha\}$ covers K, no finite subcover can cover at least one of the open balls $B\left(y_k^1, \frac{1}{2}\right)$

We will call this "uncoverable" open ball the "bad ball". Rearrange the $\{y_k^1\}$ so that the"bad ball" is labeled $B\left(y_1^1, \frac{1}{2}\right)$. Let

$$B_1 = B\left(y_1^1, \frac{1}{2}\right) \cap K.$$

Note that B_1 cannot be covered by finitely many U_α.

STEP 2: Repeat this for $\epsilon_2 = 1/4$. Again, since K is totally bounded, we can find points $y_1^2, \ldots, y_{n(2)}^2$ such that

$$K \subset \bigcup_{k=1}^{n(2)} B\left(y_k^2, \frac{1}{4}\right)$$

Since B_1 cannot be covered by finitely many U_α, no finite subcover can cover at least one of the sets $B_1 \cap B\left(y_k^2, \frac{1}{4}\right)$. Again, rearrange the $\{y_k^2\}$ so that $B\left(y_1^2, \frac{1}{4}\right)$ is the "bad ball", and let

$$B_2 = B_1 \cap B\left(y_1^2, \frac{1}{4}\right).$$

Again, B_2 cannot be covered by finitely many U_α.

STEP 3: Repeat this process with $\epsilon_n = 1/2^n$ to get a nested sequence of nonempty sets

$$K \supset B_1 \supset B_2 \supset B_3 \supset \ldots$$

such that $B_n \subset B\left(y_1^n, \frac{1}{2^n}\right)$ for some $y_1^n \in K$, and none of the B_n can be covered by finitely many U_α. We note that each set B_n is contained in a ball of radius $1/2^n$.

For each $n \in \mathbb{N}$, choose $x_n \in B_n$. Since the sets $\{B_n\}$ are nested, and each B_n is contained in a ball of radius $1/2^n$, $\{x_n\}$ is a Cauchy sequence. Since K is complete, $x_n \to x^* \in K$. Furthermore, $x^* \in B_n$ for all $n \in \mathbb{N}$.

Since $\{U_\alpha\}$ covers K, $x^* \in U_{\alpha_0}$ for some α_0. Since U_{α_0} is open and the B_n are nested, shrinking, and contain x^*, $B_n \subset U_{\alpha_0}$ for sufficiently large n, which contradicts the fact that no B_n can be covered by finitely many U_α. □

As a corollary, closed subsets of compact sets are compact.

Corollary:

Let K be a compact subset of a metric space X. If $A \subset K$ is closed, then A is compact.

Proof: Let $\{x_n\}$ be a sequence in A. Then there is a subsequence $x_{n_k} \to x^* \in K$, since K is sequentially compact. Since A is closed, $x^* \in A$ ✓ □

Next, we show that compact subsets of metric spaces are closed.

Fact:

Compact sets are closed and bounded

Proof: Let K be a compact subset of metric space X.

K **bounded:** This follows from total boundedness if you let $\epsilon = 1$

K **closed:** We will show that $X \setminus K$ is open.

Choose any $x \in X \setminus K$, i.e. $x \notin K$. For all $y \in K$, let $r(y)$ be the distance $r(y) = \frac{1}{2}d(y,x) > 0$, since $y \neq x$. The collection of open balls $\{B(y, r(y))\}_{y \in K}$ is an open cover for K, and none of them contain x by our definition of $r(y)$. By compactness, we can find a finite subcover $\{B(y_1, r(y_1)) \ldots, B(y_n, r(y_n))\}$ of K.

Let $r = \min\{r(y_1), \ldots, r(y_n)\}$. Then $B(x,r)$ does not intersect this finite subcover, which means that $B(x,r)$ lies outside of K. It follows that $B(x,r) \subset X \setminus K$, from which we conclude that $X \setminus K$ is open. □

Heine-Borel Theorem:

A subset $K \subset \mathbb{R}^n$ is compact $\iff$ K is closed and bounded.

Proof: ($\implies$) Done above

($\impliedby$) Since K is bounded, it fits inside a closed box B in $\mathbb{R}^n$. Since K is closed, and closed subsets of compact sets are compact, it suffices to show that B is compact.

Since B is bounded, it follows from the Bolzano-Weierstrass theorem that any sequence in B has a convergent subsequence, whose limit must be in B since B is closed. ✓ □

Next, we show that compactness is also a topological property, i.e. it is preserved by continuous functions.

Fact:

Let $f : (X, d_1) \to (Y, d_2)$ be continuous, and K compact in X. Then $f(K)$ is compact in Y. In other words, continuous images of compact sets are compact.

Proof: Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open cover of $f(K)$.

Since f is continuous, $f^{-1}(U_\alpha)$ is open in X, thus $\{f^{-1}(U_\alpha)\}_{\alpha \in \mathcal{A}}$ is an open cover for K.

Since K is compact, we can find a finite subcover $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ for K.

Sending the finite subcover back through f, $\{f(f^{-1}(U_1)), \ldots, f(f^{-1}(U_n))\}$ covers $f(K)$.

Since $f(f^{-1}(U_k)) \subset U_k$, $\{U_1, \ldots, U_n\}$ is a finite subcover for $f(K)$. □

The Extreme Value Theorem is a direct consequence of this.

Extreme Value Theorem:

Let $f : (X, d) \to \mathbb{R}$ continuous and $K \subset X$ compact. Then f attains an absolute maximum and an absolute minimum on K.

Proof: Since K is compact, $f(K) \subset \mathbb{R}$ is compact, thus closed and bounded by Heine-Borel. □

Definition:

A function $f : (X, d_1) \to (Y, d_2)$ is **uniformly continuous** if, for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $d_1(x, y) < \delta$, $d_2(f(x), f(y)) < \epsilon$.

The main difference between uniform continuity and continuity is that, δ depends only on ϵ, not on x

For simplicity, we will prove the uniform continuity theorem for real-valued functions, although it same result holds (with the same proof) for any pair of metric spaces.

Uniform Continuity Theorem:

Let $f : (X, d) \to \mathbb{R}$ be continuous and let $K \subset X$ compact. Then f is uniformly continuous on K.

Proof 1: Let $\epsilon > 0$. Since f is continuous on K, for every $x_0 \in K$ we can find $\delta(x_0)$ such that $|f(x) - f(x_0)| < \epsilon/2$ whenever $d(x, x_0) < \delta(x_0)$

The collection of open balls $\left\{B\left(x, \frac{\delta(x)}{2}\right)\right\}_{x \in K}$ is an open cover for K. By compactness, we can find a finite subcover. In other words, we can find points $x_1, \dots, x_n \in K$ such that

$$K \subset B\left(x_1, \delta(x_1)/2\right) \cup \cdots \cup B\left(x_n, \delta(x_n)/2\right)$$

Let $\delta = \min\{\delta(x_1)/2, \dots, \delta(x_n)/2\} > 0$. (We need this minimum to be well-defined and positive, which is why we need compactness to give us a finite subcover).

Choose any $x, y \in K$ with $d(x, y) < \delta$. We will show that $|f(x) - f(y)| < \epsilon$

Because of the finite subcover, x must be inside one of the finite set of open balls $B\left(x_k, \delta(x_k)/2\right)$. It follows that $d(x, x_k) < \delta(x_k)/2$ for some $k \in \{1, \dots, n\}$

$$d(y, x_k) \leq d(y, x) + d(x, x_k) < \delta + \delta(x_k)/2 \leq \delta(x_k),$$

since $\delta \leq \delta(x_k)/2$. Finally, by continuity of f,

$$|f(y) - f(x)| \leq |f(y) - f(x_k)| + |f(x_k) - f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

since both y and x are within a distance $\delta(x_k)$ of x_k.

Proof 2: (Will be skipped) By contradiction: Suppose the conclusion is not true. Then for a specific $\epsilon > 0$, we can find sequences $\{x_n\}, \{y_n\} \subset K$ such that $d(x_n, y_n) \to 0$, but $|f(x_n) - f(y_n)| \geq \epsilon$.

Since K is compact, thus sequentially compact, $\{x_n\}$ has a convergent subsequence $x_{n_k} \to x^*$, where $x^* \in K$, since K is closed.

Since $d(x_n, y_n) \to 0$, $y_{n_k} \to x^*$ as well.

By the triangle inequality,

$$0 < \epsilon \leq |f(x_{n_k}) - f(y_{n_k})| \leq \underbrace{|f(x_{n_k}) - f(x^*)| + |f(x^*) - f(y_{n_k})|}_{\text{both } \to 0 \text{ by continuity of } f} \to 0,$$

which is a contradiction.

2. Uniform Convergence

Our next goal is to generalize compactness, but for functions. Namely, given a sequence (f_n) of functions, we would like to extract a convergent subsequence (f_{n_k}), if that's even possible

Definition:

$f_n \to f$ **pointwise** if, for every x, we have

$$\lim_{n \to \infty} f_n(x) = f(x)$$

Example:

Let $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Notice the limit function here is discontinuous!

Definition:

$f_n \to f$ **uniformly** if for all $\epsilon > 0$ there is N such that if $n > N$ then **for all** x, we have

$$|f_n(x) - f(x)| < \epsilon$$

So for all large n, the graph of f_n is contained in an ϵ-tube around the graph of f.

Theorem: Continuity

If $f_n \to f$ uniformly and each f_n is continuous at x_0, then f is continuous at x_0.

Proof:[1] This is a typical $\frac{\epsilon}{3}$ proof:

Let $\epsilon > 0$ and x_0 be given. We need to find $\delta > 0$ such that for all x, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

STEP 1: Since $f_n \to f$ uniformly, there is N such that for all $n \geq N$ and all x, we have

[1]This proof is taken from Pugh's book, Chapter 4 Theorem 1

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

STEP 2: Since f_N is continuous at x_0 there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

STEP 3: With that δ, if $|x - x_0| < \delta$, we get

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \checkmark \end{aligned}$$

Here we used uniform convergence, continuity of f_N, and uniform convergence again, □

Theorem: Integrals

If $f_n \to f$ uniformly and each f_n is (Riemann) integrable on $[a, b]$, then so is f, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

Non-Example:

Consider $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x \leq 1 \end{cases}$$

Then $\int_0^1 f_n(x)dx = 1$ so $\lim_{n\to\infty} f_n(x)dx = 1$

But $f_n \to f = 0$ pointwise (except at $x = 0$) and $\int_0^1 f(x)dx = 0$

Finally, let's discuss differentiability, which is much more delicate!

Example:

Consider $f_n : [-1,1] \to \mathbb{R}$ defined by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$$

Then each f_n is differentiable, but f_n converges uniformly to $f(x) = |x|$, which is not differentiable!

Theorem: Differentiability

(1) Suppose f_n is differentiable on $[a,b]$ and $f_n \to f$ uniformly

(2) Moreover, suppose $f_n' \to g$ uniformly for some function g

(3) Then in fact f is differentiable and $f' = g$.

Recall that $C[a,b]$ is the set of continuous functions $f : [a,b] \to \mathbb{R}$, usually equipped with the metric

$$d(f,g) = \sup\{|f(x) - g(x)|, x \in [a,b]\}$$

Fact:

$f_n \to f$ in $C[a,b] \Leftrightarrow f_n \to f$ uniformly

Theorem:

$(C[a,b], d)$ is **complete**

Proof:[2]

STEP 1: Let f_n be a Cauchy sequence in $C[a,b]$.

Claim: For every x, $(f_n(x))$ is Cauchy (in $\mathbb{R}$)

Why? Let $\epsilon > 0$ be given, then there is N such that if $m, n > N$ then $d(f_n, f_m) < \epsilon$. With that same N, if $m, n > N$ then

$$|f_n(x) - f_m(x)| \leq \sup\{|f_n(x) - f_m(x)|, x \in [a,b]\} = d(f_n, f_m) < \epsilon$$

STEP 2: Since $(f_n(x))$ is Cauchy in $\mathbb{R}$, it converges. So for every x, it makes sense to define

$$f(x) =: \lim_{n\to\infty} f_n(x)$$

And, by definition, $f_n \to f$ pointwise

STEP 3: **Claim:** $f_n \to f$ uniformly

[2] This proof is taken from Pugh's Real Analysis book, Theorem 3 in Chapter 4

Why? Let $\epsilon > 0$ be given. Since (f_n) is Cauchy, there is N such that if $m, n > N$ then

$$d(f_n, f_m) < \frac{\epsilon}{2}$$

Take that N and let x be given

Since $f_n \to f$ pointwise, we know there is some m (depending on x) large enough such that $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ (think of it as a helper constant)

Then, if $n \geq N$, we get

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Where we have used Cauchiness and our helper constant respectrively.

Finally $f \in C[a, b]$ since the uniform limit of continuous functions is continuous □

3. Equicontinuity

Question: Is B-W still true for functions? That is: if (f_n) is a bounded sequence of functions, does it have a uniformly convergent subsequence (f_{n_k})?

Unfortunately the answer is no ☹

(See example from next time)

That said, the answer is yes if we add an additional assumption which is equicontinuity:

Definition:

A sequence (f_n) is (uniformly) **equicontinuous** if for all $\epsilon > 0$ there is $\delta > 0$ such that for all n and all x, y, if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$

Equicontinuity just means that δ doesn't depend on n, it's the same for all n.