LECTURE: THE ARZELÀ-ASCOLI THEOREM

1. Equicontinuity

Question: Is B-W still true for functions? If (f_n) is a bounded sequence of functions, does it have a uniformly conv subsequence (f_{n_k})?

Unfortunately the answer is no ☹

Definition:

(f_n) is **bounded** if there is M such that for all n and x we have

$$|f_n(x)| \leq M$$

Non-Example:

Consider the sequence $f_n(x) = \sin(nx)$ on $[0, 2\pi]$

Then $|f_n(x)| = |\sin(nx)| \leq 1$, so f_n is bounded.

Suppose f_n had a uniformly conv subsequence $f_{n_k} \to f$ for some f.

$$\text{Then } \lim_{k\to\infty} \sin(n_k x) - \sin(n_{k+1}x) = f(x) - f(x) = 0$$

Squaring this, we get

$$\lim_{k\to\infty} \left(\sin(n_k x) - \sin(n_{k+1}x)\right)^2 = 0$$

Therefore

$$\lim_{k\to\infty}\int_0^{2\pi}\left(\sin(n_k x)-\sin(n_{k+1}x)\right)^2 dx=\int_0^{2\pi}\underbrace{\lim_{k\to\infty}\left(\sin(n_k x)-\sin(n_{k+1}x)\right)^2}_{0} dx=0$$

The passage of the limit inside the integral is justified by the Bounded Convergence Theorem

However, if you actually calculate the integral using double angle formulas, you get for all k

$$\int_0^{2\pi}\left(\sin(n_k x)-\sin(n_{k+1}x)\right)^2 dx=2\pi\nrightarrow 0$$

Which is a contradiction

That said, the answer is **YES** if you add an additional hypothesis:

Definition:

A sequence (f_n) is (uniformly) **equicontinuous** if for all $\epsilon>0$ there is $\delta>0$ such that <u>for all n</u> and all x, y, if $|x-y|<\delta$, then $|f_n(x)-f_n(y)|<\epsilon$

Equicontinuity just means that δ doesn't depend on n, it's the same for all n.

2. Arzelà-Ascoli Theorem

We are ready to state and prove the celebrated Arzelà-Ascoli Theorem:

Arzelá-Ascoli Theorem:

If (f_n) is a bounded and equicontinuous sequence in $C[a,b]$, then (f_n) has a uniformly convergent subsequence.

Proof:[1]

STEP 1: Fix an enumeration $\{x_1, x_2, \dots\}$ of all the rational numbers in $[a,b]$

Consider $f_n(x_1)$. This is a bounded sequence of real numbers since (f_n) is bounded, so by B-W, there is a convergent subsequence $f_{n_k}(x_1)$

Notation:

$$\begin{aligned} f_{0,k} =& f_k \text{ original sequence} \\ f_{1,k} =& f_{n_k} \text{ subsequence} \\ f_{2,k} =& \text{ sub-subsequence} \\ f_{m,k} =& \text{ sub-sub... sequence} \end{aligned}$$

m is the "depth" of the sequence, and k is the term of the sequence

Since $f_{1,k}(x_2)$ is bounded, there is a sub-subsequence $f_{2,k}$ such that $f_{2,k}(x_2)$ converges. Notice $f_{2,k}$ converges as x_1 as well. So $f_{2,k}$ converges at x_1 and x_2

That way we obtain a tower of subsequences

$$f_n \supseteq f_{1,k} \supseteq f_{2,k} \supseteq \dots$$

[1]The proof is taken from this Wikipedia article, as well as from Theorem 14 in Chapter 4 of Pugh's book

Such that $f_{m,k}$ converges at $x_1, x_2, \ldots, x_m$

STEP 2: Consider the diagonal subsequence $g_m =: f_{m,m}$ which is the m-th term of the m-th subsequence.

By construction, g_m converges at every rational point.

Claim: (g_m) converges uniformly

Then we would be done because then (g_m) is a subsequence of (f_n) that converges uniformly.

STEP 3:

Proof of Claim: We will show that (g_m) is Cauchy.

Here is where equicontinuity kicks in:

Let $\epsilon > 0$ be given.

By equicontinuity there is $\delta > 0$ such that for all x, y and all m:

$$|x - y| < \delta \Rightarrow |g_m(x) - g_m(y)| < \frac{\epsilon}{3}$$

Intuitively: Rational points are good (because g_m converges on them) and δ is good (because of continuity), it makes sense to cover $[a, b]$ with balls centered at rational points and radius δ:

Consider the balls (intervals) $B(x_1, \delta), B(x_2, \delta), \ldots$ They cover $[a, b]$ so by compactness there is a finite sub-cover, which we'll relabel as $B(x_1, \delta), B(x_2, \delta), \ldots, B(x_I, \delta)$.

Since $g_m(x_i)$ converges for each x_i as above, it is Cauchy, so there is N such that for all $m, n > N$ and all $i = 1, 2, \dots, I$

$$|g_m(x_i) - g_n(x_i)| < \frac{\epsilon}{3}$$

STEP 4: Now we're ready to conclude!

With the same N, if $m, n > N$ and $x \in [a, b]$, choose x_i as above such that $|x_i - x| < \delta$ (can do that by def of a cover) then

$$\begin{aligned} |g_m(x) - g_n(x)| &\leq |g_m(x) - g_m(x_i)| + |g_m(x_i) - g_n(x_i)| + |g_n(x_i) - g_n(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \checkmark \end{aligned}$$

(By equicontinuity, Cauchiness, and equicontinuity)

Note: The same proof works for $C(K)$ where K is compact. In that case we use that K is separable, i.e. it has a countable dense subset. That dense subset serves as an anlog of rational numbers.

We also have a partial converse to the Arzela-Ascoli theorem.

Theorem:

If $A \subset C(K)$ is compact, then it is bounded and equicontinuous.

Proof:

STEP 1: Let $\epsilon > 0$. Using the definition of totally bounded, there exists a finite collection of functions $f_1, \dots, f_n \in A$ such that

$$A \subset \bigcup_{k=1}^{n} B(f_k, \epsilon)$$

Here both the f_i and n depend on ϵ

STEP 2: Choose any $f \in A$. Then $f \in B(f_k, \epsilon)$ for some $k \in \{1, \dots, n\}$. In particular, this means that $\sup_{x \in K} |f(x) - f_k(x)| < \epsilon$.

Boundedness:

$$\begin{aligned} \sup_{x \in K} |f(x)| &\leq \sup_{x \in K} |f(x) - f_k(x)| + \sup_{x \in K} |f_k(x)| \\ &\leq \epsilon + \left(\max_{j=1,\dots,n} \sup_{x \in K} |f_j(x)| \right) =: M < \infty \end{aligned}$$

where $\sup_{x \in K} |f_j(x)|$ is finite by the extreme value theorem, and we are taking the maximum over a finite set. Since M is independent of f, we conclude that A is uniformly bounded.

STEP 3: Equicontinuity:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &\leq 2\epsilon + \max_{j=1,\dots,n} |f_j(x) - f_j(y)|. \end{aligned}$$

STEP 4: Since K is compact, each function $f_j(x)$ is uniformly continuous on K. This means that for all $j = 1, \dots, n$, we can find $\delta_j > 0$ such that if $|x - y| < \delta_j$ then $|f_j(x) - f_j(y)| < \epsilon$.

Let $\delta = \min\{\delta_1, \dots, \delta_n\} > 0$.

Then, if $|x - y| < \delta$ we have $\max_{j=1,\dots,n} |f_j(x) - f_j(y)| < \epsilon$

Combining the previous steps, if $|x - y| < \delta$, then

$$|f(x) - f(y)| < 3\epsilon.$$

Since δ is independent of f, we conclude that A is equicontinuous. $\square$

3. Lipschitz Continuity

In practice it's a pain to apply Arzelà-Ascoli directly, especially when you need to check equicontinuity. Fortunately, there is a nice shortcut which is called Lipschitz Continuity. This concept appears in many different contexts, especially in the theory of ODE

Definition:

$f : (X, d) \to \mathbb{R}$ is **Lipschitz** if there is $L > 0$ such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq Ld(x, y).$$

In other words, the difference in outputs is not much bigger than the difference in inputs. Intuitively, the Lipschitz constant L puts a bound on the slopes of all of the possible secant lines of f. A Lipschitz function is "nice" in the sense that it "does not change too fast".

Theorem:

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and $|f'(x)| \leq L$ for all x. Then f is Lipschitz continuous, with Lipschitz constant L.

Proof: Let x and y be given. Then by the mean value theorem, there exists a point c between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

This implies

$$|f(x) - f(y)| = |f'(c)||x - y| \leq L|x - y| \checkmark \quad \square$$

Fact:

Let $K \subset \mathbb{R}^d$ be compact and $A \subset C(K)$. If every function in A is Lipschitz continuous with the same Lipschitz constant L, then A is equicontinuous.

Proof: Let $\epsilon > 0$, and choose $\delta = \epsilon/2L$. Then for all $x, y \in K$ with $|x - y| < \delta$ and for all $f \in A$,

$$|f(x) - f(y)| \leq L|x - y| \leq L\delta = \frac{\epsilon}{2} < \epsilon.$$

The same result holds for Hölder continuous functions:

Definition:

A function $f : \mathbb{R}^d \to \mathbb{R}$ is **Hölder continuous** with exponent $\alpha \in (0, 1]$ if there is $C > 0$ such that for all $x, y \in X$,

$$|f(x) - f(y)| \leq C|x - y|^\alpha.$$

The case $\alpha = 1$ is Lipschitz continuity.

We mentioned above that the mean value theorem implies that if $f : [a, b] \to \mathbb{R}$ is continuously differentiable, f is Lipschitz with Lipschitz constant $\max_{x \in [a,b]} |f'(x)|$. Unfortunately, this does not work in higher dimensions, since there is no $n-$dimensional analogue to the mean value theorem.

However, we can obtain a similar result, if f is continuously differentiable on a compact and convex set.

Definition:

A set E in $\mathbb{R}^d$ is **convex** if for all $x, y \in E$, the line segment joining them is also in E. In other words, for all $x, y \in E$,

$$tx + (1-t)y \in E \qquad\qquad t \in [0,1].$$

We then have the following fact:

Fact:

Let $K \subset \mathbb{R}^d$ convex and compact, and $K \subset U$, where U is open. Let $f : U \to \mathbb{R}$ be continuously differentiable, and let $\sup_{x \in K} \|Df(x)\| = L$.

Then $f : K \to \mathbb{R}$ is Lipschitz with constant L.

Proof: Let $x, y \in K$. Then by the Fundamental Theorem of Calculus,

$$\begin{aligned} f(x) - f(y) &= \int_0^1 \frac{d}{dt} f\left(tx + (1-t)y\right) dt \\ &= \int_0^1 Df(tx + (1-t)y) \cdot (x-y) dt \\ &= \left(\int_0^1 Df(tx + (1-t)y) dt \right) \cdot (x-y), \end{aligned}$$

Taking absolute values, we have

$$\begin{aligned}|f(x)-f(y)| &\leq \left|\int_0^1 Df(tx+(1-t)y)dt\right| |x-y| \\ &\leq \left(\int_0^1 \underbrace{\|Df(tx+(1-t)y)\|}_{\leq L \text{ since } tx+(1-t)y \in K} dt\right) |x-y| \\ &\leq L|x-y|.\end{aligned}$$

4. ODE Existence Theorem

Here is a nice application of the Arzela-Ascoli theorem to ODEs. Let's first look at a couple of examples to see what can go wrong

Example 1:

$$\begin{cases} \frac{du}{dt} &= ku \\ u(0) &= u_0 \end{cases}$$

Then $u(t) = u_0 e^{kt}$, so there is a unique solution existing for all time. There is nothing wrong with this example

Example 2:

$$\begin{cases} \frac{du}{dt} &= u^2 \\ u(0) &= 1 \end{cases}$$

By separation of variables, $u(t) = \frac{1}{1-t}$

Notice $u(t) \to \infty$ as $t \to 1$ from the left. In other words, the solution blows up to infinity in finite time

Example 3:

$$\begin{cases} \dfrac{du}{dt} = \sqrt{u} \\ u(0) = 0 \end{cases}$$

By separation of variables, one solution is $u(t) = \frac{1}{4}t^2$

But by inspection, $u(t) = 0$ is another solution.

Even worse, there is actually an infinite number of solutions, which can be written as

$$u(t) = \begin{cases} 0 & t \leq C \\ \frac{1}{4}(t-C)^2 & t > 0 \end{cases}$$

So uniqueness fails pretty badly!

Given those examples, the best we can hope for (at least) is to prove local existence, that is existence near the initial condition. The most basic result is due to Peano, where only continuity is assumed.

Cauchy-Peano Existence Theorem:

Consider the initial value problem on $\mathbb{R}$

$$\begin{cases} \dfrac{du}{dt} = f(t, u) \\ u(t_0) = u_0 \end{cases}$$

If f is continuous in a neighborhood of (t_0, u_0), then there exists at least one solution $u(t)$ defined in a neighborhood of t_0.

Note: The same proof works for $\mathbb{R}^n$ and by translation you can assume $u(0) = 0$

(We will prove this next time)