LECTURE: APPLICATIONS TO ODE

1. ODE Existence Theorem

Cauchy-Peano Existence Theorem:

Consider the initial value problem on $\mathbb{R}$

$$\begin{cases} \dfrac{du}{dt} = f(t,u) \\ u(0) = 0 \end{cases}$$

If f is continuous in a neighborhood of $(0,0)$, then there exists at least one solution $u(t)$ defined in a neighborhood of t_0.

Proof:

STEP 1: Since f is continuous in a neighborhood of $(0,0)$, f is continuous on a box $B = [-R,R] \times [-R,R]$ for some $R > 0$, and since boxes are compact, there is $M \geq 1$ such that $|f(t,u)| \leq M$ on B

STEP 2: Rewrite the problem in integral form. This is useful because integrals are easier to deal with than derivatives

$$\int_0^t \frac{d}{ds} u(s) ds = \int_0^t f(s, u(s)) ds$$

$$u(t) = \int_0^t f(s, u(s)) ds$$

Main Idea: Use Euler's method on a very fine grid.

STEP 3: Euler's Method

Let $\boxed{T = \frac{R}{M}}$

Given $n \in \mathbb{R}$ let $\boxed{h_n = \frac{T}{n}}$ be the mesh size for the time grid $[-T, T]$ so that the grid for t is given by

$$\begin{aligned} &[-t_n^n, -t_{n-1}^n, \dots, -t_1^n, 0, t_1^n, t_2^n, \dots, t_n^n] \\ =&[-nh_n, -(n-1)h_n, \dots, -h_n, 0, h_n, 2h_n, \dots, nh_n]. \end{aligned}$$

We start with the initial condition $u_0^n = 0$, and then we compute u_m^n on the rest of the grid using the forward Euler method:

$$\begin{aligned} u_0^n &= 0 \\ u_1^n &= 0 + f(0,0)h_n \\ u_2^n &= u_1^n + f(t_1^n, u_1^n)h_n \\ &\vdots \\ u_{m+1}^n &= u_m^n + f(t_m^n, u_m^n)h_n \\ &\vdots \end{aligned}$$

And similarly for going backwards in t

Define $u_n(t)$ to be the piecewise linear interpolation of these grid values, i.e. "connect the dots" by joining the points (t_m^n, u_m^n) with line segments.

Notice that by construction we have

$$u_n'(t) = f(t_m^n, u_m^n) \text{ on } (t_m^n, t_{m+1}^n)$$

It follows from the bound on f that $|u_n'(t)| \leq M$ except at the mesh points

Main Idea: Apply Arzelà-Ascoli to $\{u_n(t)\}$ to extract a uniformly convergent subsequence.

STEP 4: Show $\{u_n(t)\}$ is bounded and equicontinuous

Boundedness: For each Euler step, we have

$$|u_{m+1}^n - u_m^n| = |f(t_m^n, u_m^n)h_n| \leq M h_n$$

Since the function $u_n(t)$ involves (at most) n Euler steps in each direction, and $u_n(t)$ is linear between these steps, we have

$$|u_n(t)| \leq nMh_n = MT \checkmark$$

Equicontinuity: Since $u_n(t)$ is differentiable almost everywhere, we have for $-T \leq s, t \leq T$,

$$|u_n(t) - u_n(s)| \leq \int_s^t |u_n'(r)| dr \leq \int_s^t M dr = M|t-s| \checkmark$$

Hence by the Arzela-Ascoli theorem, $\{u_n(t)\}$ has a uniformly convergent subsequence $\{u_{n_k}(t)\} = \{v_k(t)\}$ which converges to some $u(t)$

Claim: The limit $u(t)$ solves our ODE

STEP 5: Proof of Claim:

$$
\begin{aligned}
&\left|u(t)-\int_0^t f(s,u(s))ds\right| \\
\leq &|u(t)-v_k(t)| + \left|v_k(t)-\int_0^t f(s,v_k(s))ds\right| + \left|\int_0^t f(s,v_k(s))ds - \int_0^t f(s,u(s))ds\right| \\
= &A+B+C
\end{aligned}
$$

Study of A: This term goes to 0 as $k\to\infty$ because $v_k\to u$ uniformly

Study of C:

$$
\left|\int_0^t f(s,v_k(s))ds - \int_0^t f(s,u(s))ds\right| \leq \int_0^t |f(s,v_k(s))ds - f(s,u(s))|ds,
$$

This also goes to 0 by the uniform convergence of $v_k(t)$ and because f is uniformly continuous on $[-T,T]$

STEP 6: Study of B

Let's focus on the mesh points of $v_k(t)$ which we'll label as t_j^k

Let $t\in[0,T]$, then $t\in\left[t_m^k, t_{m+1}^k\right]$ for some m (If t is one of the grid points then $t=t_{m+1}^k$)

Let v_j^k be the values of $v_k(t)$ on the grid t_j^k that is $\boxed{v_j^k = v_k\left(t_j^k\right)}$

Then write $v_k(t)$ as the following telescoping sum (recall $v_k(0)=0$)

$$
v_k(t) = \sum_{j=0}^{m-1}\left(v_{j+1}^k - v_j^k\right) + \left(v_k(t)-v_m^k\right)
$$

Then we get

$$
\begin{aligned}
B =& v_k(t) - \int_0^t f(s, v_k(s))ds \\
=& \sum_{j=0}^{m-1} (v_{j+1}^k - v_j^k) - \sum_{j=0}^{m-1} \int_{t_j^k}^{t_{j+1}^k} f(s, v_k(s))ds + (v_k(t) - v_m^k) - \int_{t_m^k}^t f(s, v_k(s))ds \\
=& \sum_{j=0}^{m-1} \int_{t_j^k}^{t_{j+1}^k} v_k'(s, v_k(s))ds - \sum_{j=0}^{m-1} \int_{t_j^k}^{t_{j+1}^k} f(s, v_k(s))ds + \int_{t_m^k}^t v_k'(s, v_k(s))ds - \int_{t_m^k}^t f(s, v_k(s)) \\
=& \sum_{j=0}^{m-1} \int_{t_j^k}^{t_{j+1}^k} (f(t_j^k, v_j^k) - f(s, v_k(s)))ds + \int_{t_m^k}^t f(t_m^k, v_m^k) - f(s, v_k(s)))ds
\end{aligned}
$$

Taking absolute values, we obtain

$$
\begin{aligned}
&\left| v_k(t) - \int_0^t f(s, v_k(s))ds \right| \\
\leq& \sum_{j=0}^{m-1} \int_{t_j^k}^{t_{j+1}^k} |f(t_j^k, v_j^k) - f(s, v_k(s))|ds + \int_{t_m^k}^t |f(t_m^k, v_m^k) - f(s, v_k(s))|ds \\
\leq& \sum_{j=0}^{m} \int_{t_j^k}^{t_{j+1}^k} |f(t_j^k, v_j^k) - f(s, v_k(s))|ds.
\end{aligned}
$$

STEP 7: All that remains is to estimate this integral!

Let $\epsilon > 0$. Then by uniform continuity of f there is $\delta > 0$ such that if $|s - t| < \delta$ and $|u - v| < \delta$, then $|f(s, u) - f(t, v)| < \epsilon$.

Then let k sufficiently large so that $h_{n_k} < \delta / M$.

Then, for all $s \in [t_m^k, t_{m+1}^k]$,

$$|s - t_m^k| \leq |t_{m+1}^k - t_m^k| < \delta$$

$$\text{and } |v_k(s) - v_m^k| \leq |v_{m+1}^k - v_m^k| \leq Mh_{n_k} < \delta,$$

since $v_k(s)$ is a piecewise interpolation between v_k^k and v_{m+1}^k.

For all the integrands involved in the sum, we have the bound

$$|f(t_j^k, v_j^k) - f(s, v_k(s))| \leq \epsilon.$$

Putting all of this together, we have

$$\begin{aligned}
&\left| v_k(t) - \int_0^t f(s, v_k(s)) ds \right| \\
\leq &\sum_{j=0}^{m} \int_{t_j^k}^{t_{j+1}^k} \epsilon ds = \epsilon \sum_{j=0}^{m} (t_{j+1}^j - t_j^k) ds \\
\leq &\epsilon \sum_{j=0}^{m} h_{n_k} ds \\
\leq &\epsilon (m+1) h_{n_k} \\
\leq &\epsilon n_k \left(\frac{T}{n_k} \right) \\
= &\epsilon T \checkmark
\end{aligned}$$

The following inequality is useful for ODE, in particular when proving uniqueness of solutions:

2. Grönwall's Inequality

Grönwall's Inequality:

Let $u(t)$ and $g(t)$ be non-negative, real-valued functions defined on $t \in [0, T]$. Suppose that for some $C \geq 0$ we have

$$u(t) \leq C + \int_0^t g(s)u(s)ds$$

$$\text{Then } u(t) \leq C \exp\left(\int_0^t g(s)ds\right)$$

In particular, if $C = 0$ then $u \equiv 0$ on $[0, T]$

Proof: We first consider the case where $C > 0$. Let

$$v(t) =: C + \int_0^t g(s)u(s)ds.$$

Then $u(t) \leq v(t)$ (by assumption), and $v(t) \geq C > 0$

Differentiating v with respect to t, we obtain

$$v'(t) = g(t)u(t) \leq g(t)v(t).$$

Since $v(t) > 0$, we can divide by $v(t)$ to obtain

$$\frac{v'(t)}{v(t)} \leq g(t)$$

Next, we integrate the function g to obtain

$$\int_0^t g(s)ds \geq \int_0^t \frac{v'(s)}{v(s)}ds = \int_0^t \frac{d}{ds}\left(\ln v(s)\right)ds = \ln\frac{v(t)}{v(0)} = \ln\frac{v(t)}{C}.$$

Finally, we exponentiate both sides and multiply by C to get

$$u(t) \leq v(t) \leq C \exp\left(\int_0^t g(s)ds\right) \checkmark$$

If $C = 0$, then for any $\epsilon > 0$,

$$u(t) \leq \epsilon + \int_0^t g(s)u(s)ds$$

By the first result (taking $C = \epsilon$), we have

$$u(t) \leq \epsilon \exp\left(\int_0^t g(s)ds\right) \leq \epsilon \exp\left(\int_0^T g(s)ds\right) \leq M\epsilon,$$

Hence $u \equiv 0$ since ϵ is arbitrary □

3. Uniqueness

We can use the Gronwall inequality to prove the uniqueness of solutions to the initial value problem $\frac{du}{dt} = f(t,u)$ in the case where the function f is Lipschitz continuous.

Local Uniqueness:

Consider the initial value problem on $\mathbb{R}^n$

$$\begin{cases} \dfrac{du}{dt} = f(t,u) \\ u(t_0) = u_0 \end{cases}$$

Suppose that f is Lipschitz continuous in u in a neighborhood of (t_0, u_0), with a Lipschitz constant L independent of t.

Then there is a *unique* sol $u(t)$ defined in a neighborhood of t_0.

Proof: WLOG $t_0 = 0$

Local existence follows from the Cauchy-Peano existence theorem.

For uniqueness, suppose that two functions $u_1(t)$ and $u_2(t)$ are local solutions to the initial value problem, both of which exist on a time interval $[-T, T]$.

If necessary, shrink T so that the Lipschitz condition holds on all of $[-T, T]$.

We will show that $u_1(t) = u_2(t)$ on $[-T, T]$.

Let $u(t) = u_1(t) - u_2(t)$.

Since both u_1 and u_2 solve the integrated form of the ODE,

$$\begin{aligned} |u(t)| &= \left| \left(u_0 + \int_0^t f(s, u_1(s)) ds \right) - \left(u_0 + \int_0^t f(s, u_2(s)) ds \right) \right| \\ &= \int_0^t |f(s, u_1(s)) - f(s, u_2(s))| ds \leq \int_0^t L|u_1(s) - u_2(s)| ds \\ &= 0 + \int_0^t Lu(s) ds \end{aligned}$$

We have satisfied the conditions of the Gronwall inequality with $g(t) = L$ and $C = 0$. It follows that $u(t) = 0$ for $t \in [0, T]$, from which we conclude that $u_1(t) = u_2(t)$ for $t \in [0, T]$. We can similarly obtain the result for $t \in [-T, 0]$ □