LECTURE: INTEGRATION AND ODE

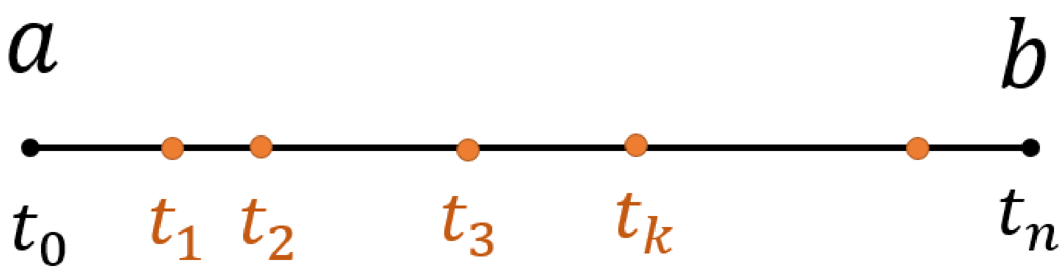
1. The Darboux Integral

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The Darboux integral is defined as follows.

STEP 1: First, partition the domain $[a,b]$. Let P be the partition

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

where n is arbitrary.

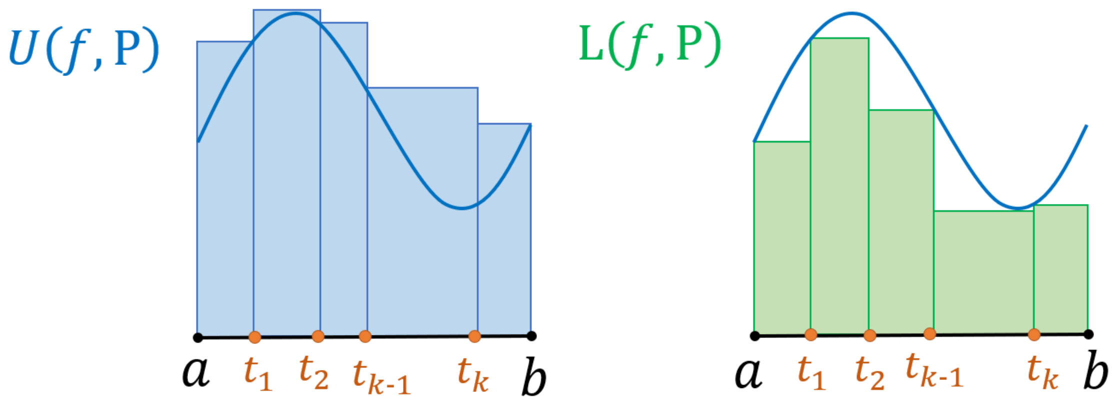


(The picture uses t_k instead of x_k)

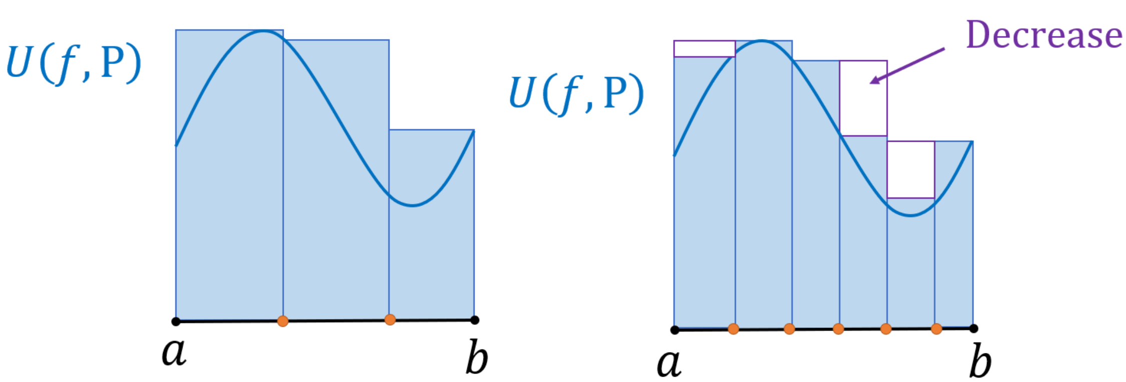
STEP 2: Define the upper and lower sums on P by

$$U(f,P) = \sum_{j=0}^{n-1} (x_{j+1} - x_j) \sup_{x \in [x_j, x_{j+1}]} f(x)$$

$$L(f,P) = \sum_{j=0}^{n-1} (x_{j+1} - x_j) \inf_{x \in [x_j, x_{j+1}]} f(x).$$



Notice that refining a partition causes $U(f,P)$ to decrease and $L(f,P)$ to increase



STEP 3:

Definition:

The function f is Darboux integrable on $[a,b]$ if

$$\inf_P U(f,P) = \sup_P L(f,P)$$

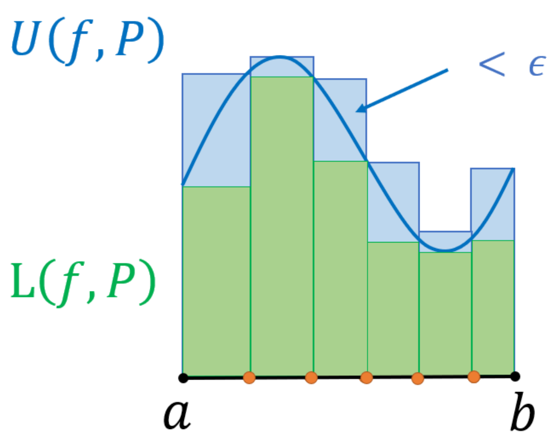
If this holds, we denote the Darboux integral by $\int_a^b f(x)dx$.

A convenient integrability criterion is the following:

Darboux Integrability Criterion:

f is Darboux integrable on $[a,b]$ if, for every $\epsilon > 0$, we can find a partition P such that

$$U(f,P) - L(f,P) < \epsilon.$$



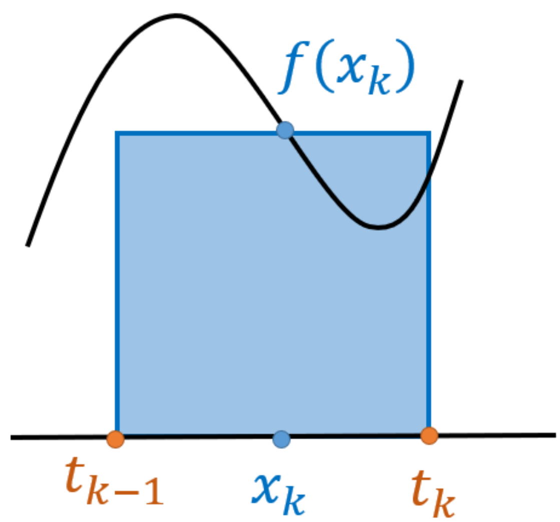
2. The Riemann Integral

This is in contrast with the Riemann Integral:

Choose a tagged partition (P, t) of $[a, b]$ that is a partition P given by

$$P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

together with n points $\{t_0, t_1, \ldots, t_{n-1}\}$, with one selected from each subinterval of the partition, i.e. $t_j \in [x_j, x_{j+1}]$. Common choices for the tags t_i are the left endpoints, the right endpoint, and the midpoints of the partition intervals, although the choice of tags does not matter from a theoretical standpoint.



(The picture has x_i and t_i switched)

The mesh size of the partition is the maximum length of the partition subintervals, i.e.

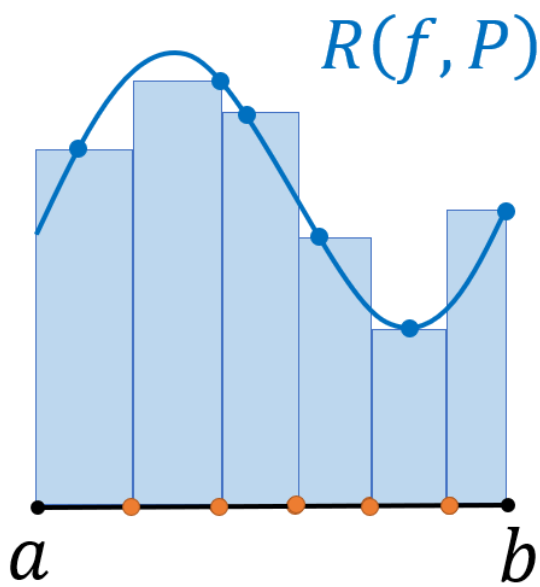
$$\Delta P = \max_{j=0,\ldots,n-1} (x_{j+1} - x_j).$$

Typically, each partition subinterval is chosen to be the same size, although this need not be the case.

Definition:

The **Riemann sum** corresponding to (P, t) is

$$R(f, P) = \sum_{j=0}^{n-1} (x_{j+1} - x_j) f(t_j).$$



Definition:

The function f is Riemann integrable with integral S if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|R(f, P) - S| < \epsilon$$

for all tagged partitions (P, t) with mesh size $\Delta P < \delta$

It can be shown that Riemann integrability and Darboux integrability are equivalent.

3. Advantages and Disadvantages

Advantages of Riemann Integral:

- You can compute them exactly with the fundamental theorem of calculus (as long as you can find an antiderivative!)
- The definition is intuitive, and captures the idea of finding the area under a curve by successive approximation.
- The approximating Riemann sums are easy to compute numerically.
- Many useful classes of functions are Riemann integrable:
  (1) Continuous functions on $[a, b]$.
  (2) Bounded functions on $[a, b]$ which are continuous except at a finite number of points.
  (3) Bounded, monotonic functions on $[a, b]$.

Disadvantages of Riemann Integral:

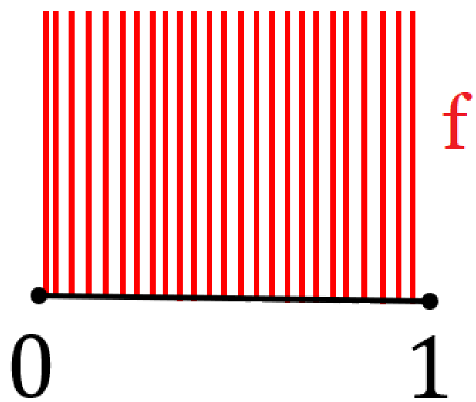
- It is difficult to extend to domains that are not open subsets of $\mathbb{R}^n$.
- It is difficult to extend to unbounded domains, such as all of $\mathbb{R}$. As in calculus class, you can define an "improper integral" as the limit of integrals on bounded intervals, although the best way to do this is not always clear.

- Some important functions are not Riemann Integrable

Non-Example 1:

Consider the following function on $[0, 1]$:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$



Then $\sup f = 1$ and $\inf f = 0$ on each sub-piece

$$\begin{aligned} U(f,P) = &\sum_{k=1}^{n} 1(x_k - x_{k-1}) \\ =&x_n - x_0 \\ =&1 - 0 \\ =&1 \end{aligned}$$

Since $U(f, P) = 1$ for all P, $\inf_P U(f, P) = 1$

$$L(f,P) = \sum_{k=1}^{n} 0(x_k - x_{k-1}) = 0$$

Therefore $\sup L(f,P) = 0$

Since $0 \neq 1$, f is not Darboux integrable

- The limit of a sequence of Riemann integrable functions is not necessarily Riemann integrable.

Non-Example 2:

Enumerate the rational numbers in $[0,1]$ as $r_1, r_2, \cdots$, and define the sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & x \in r_1, \dots, r_n \\ 0 & \text{otherwise.} \end{cases}$$

Since f_n has only a finite number of discontinuities, f_n is Riemann integrable with

$$\int_0^1 f_n(x)dx = 0.$$

For every $x \in [0,1]$, $f_n(x) \to \chi_{\mathbb{Q}}(x)$, but $\chi_{\mathbb{Q}}(x)$ is not Riemann integrable on $[0,1]$.

A final disadvantage (and perhaps the most important one), is that it is hard to find good criteria that allow us to exchange limits and integration. We would like to find conditions for which

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx,$$

i.e. the limit of the integrals is the integral of the limit. The best we can do, in general, is if we have a uniformly convergent sequence of functions on a bounded interval.

Theorem:

For all $n \in \mathbb{N}$, let $f_n : [a, b] \to \mathbb{R}$ be a Riemann integrable functions, and suppose the sequence of functions $\{f_n\}$ converges uniformly to f. Then f is Riemann integrable, and

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof: Let $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ so that

$$f_n(x) - \epsilon_n \le f(x) \le f_n(x) + \epsilon_n \text{ on } [a, b]$$

Since $\{f_n\}$ converges to f uniformly we have $\epsilon_n \to 0$

Then we have

$$\begin{aligned}\int_a^b f_n(x)dx - (b-a)\epsilon_n = \int_a^b (f_n(x) - \epsilon_n)dx &= \sup_P L_P(f_n(x) - \epsilon_n)\\ &= \sup_P \sum_{j=0}^{n-1}(x_{j+1} - x_j) \inf_{x\in[x_j,x_{j+1}]}(f(x) - \epsilon_n)\\ &\le \sup_P \sum_{j=0}^{n-1}(x_{j+1} - x_j) \inf_{x\in[x_j,x_{j+1}]} f(x)\\ &= \sup_P L_P(f),\end{aligned}$$

where P is an arbitrary partition of $[a, b]$. Similarly,

$$\inf_P U_p(f) \leq \int_a^b f_n(x)dx - (b-a)\epsilon_n$$

Putting these together

$$\int_a^b f_n(x)dx - (b-a)\epsilon_n \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq \int_a^b f_n(x)dx - (b-a)\epsilon_n.$$

This rearranges to

$$0 \leq \inf_P U(f, P) - \sup_P L_P(f) \leq 2(b-a)\epsilon_n \to 0 \quad \square$$

Non-Example 3:

To show that we really need uniform convergence in the above, consider

$$f_n(x) = nx\left(1-x^2\right)^n$$

Then can check that $f_n \to 0$ pointwise on $[0, 1]$

Now *if* the above result were true, then we would have $\int_0^1 f_n(x)dx \to \int_0^1 0dx = 0$, but using the $u-$sub $u = 1 - x^2$, it follows that

$$\int_0^1 f_n(x)dx = \int_0^1 nx\left(1-x^2\right)^n dx = \frac{n}{2n+2} \to \frac{1}{2} \neq 0$$

Moreover, the main issue is that uniform convergence is, in a sense, too strong condition.

Example 4:

Consider the sequence of functions $f_n(x) = x^n$ on $[0,1]$. For the limit, $f_n(x) \to f(x)$, where

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1, \end{cases}$$

but this convergence is not uniform.

However, we still have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \frac{1}{1+n} = 0 = \int_0^1 f(x)dx.$$

Even though the convergence is not uniform, the limit of the integrals is equal to the integral of the limit. Something else must be going on, and we would like to have a theory which explains this case.

Note: The Lebesgue integral, which we'll define later, resolves most of the weaknesses of Riemann integral. The main drawback is that, in most cases you cannot actually compute an integral using the Lebesgue formulation, and you have to fall back on the fundamental theorem of calculus from Riemann integration theory. Luckily, if a function is Riemann integrable, it is also Lebesgue integrable, and the two integrals are the same!

4. More ODE Existence-Uniqueness

Consider once again the initial value problem (IVP) on $\mathbb{R}^n$

$$\begin{cases} \dfrac{du}{dt} = f(u) \\ u(0) = u_0 \end{cases}$$

Question: How does the solution u depend on u_0?

Definition:

$f : \mathbb{R}^n \to \mathbb{R}^n$ is **locally Lipschitz** if for every $x \in \mathbb{R}^n$ there are $\delta > 0$ and L such that for all $y, z \in B(x, \delta)$

$$|f(y) - f(z)| \leq L\,|y - z|$$

Picard-Lindelöf (general version):

Suppose that f is **locally Lipschitz** and consider the IVP above.

Then for every $\overline{u} \in \mathbb{R}^n$ there is $\delta > 0$ and a time interval $[-r, r]$ such that:

(1) For each initial condition $u_0 \in B(\overline{u}, \delta)$ the IVP has a unique solution $u(t; u_0)$ on $[-r, r]$

(2) The map $u_0 \mapsto u(\cdot; u_0)$ is Lipschitz in u_0

(3) If f is C^k for $k \geq 1$, then

  (a) The solution $u(t; u_0)$ is C^{k+1} in t

  (b) The map $u_0 \mapsto u(\cdot; u_0)$ is C^k in u_0

Let's now look at what happens when a solution to an ODE approaches the boundary of the region where the solution exists. The following theorem shows that such a solution must blow up at the boundary

Blow-up at boundary:

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $u(t)$ satisfies the IVP above on an interval $[0, T)$

Suppose there is no solution to the IVP on the interval $[0, T + \epsilon)$ for any $\epsilon > 0$

Then $|u(t)| \to \infty$ as $t \to T$. That is, the solution blows up at the boundary of the time interval

Proof: We prove this by contradiction. The main idea is that if we assume $u(t)$ remains bounded, we can construct a solution to the IVP which exists on a larger time interval, i.e. at time $t > T$

STEP 1: Suppose this is not true. Then there is some sequence of times $\{t_n\}$ with $t_n \nearrow T$ such that $\{u(t_n)\}$ remains bounded, i.e. there exists a constant K such that $|u(t_n)| \leq K$ for all n

Since $\{u(t_n)\}$ is a bounded sequence in $\mathbb{R}^n$, by the Bolzano-Weierstrass theorem, it has convergent subsequence. Passing to this subsequence if needed, we may assume that $u(t_n) \to \overline{u}$

STEP 2: Now consider the same IVP but with $u(0) = \overline{u}_0$ where $\overline{u}_0$ is close to $\overline{u}$.

By the general Picard-Lindelöf there is $\delta > 0$ and an interval $[-r, r]$ such that for all initial conditions $\overline{u}_0 \in B(\overline{u}, \delta)$ there is a unique solution $u(t)$ for $t \in [-r, r]$ with $u(0) = \overline{u}_0$.

Since f does not depend on t, we can translate these unique solutions in time, i.e shift them to different starting times. In other words, for each $\overline{u}_0 \in B(\overline{u}, \delta)$, there is a family of unique solutions $u(t;\tau)$ on the interval $[\tau - r, \tau + r]$ with $u(\tau;\tau) = \overline{u}_0$

Since $t_n \nearrow T$ and $u(t_n) \to \overline{u}$, choose an integer m sufficiently large so that $t_m > T - r/2$ and $u(t_m) \in B(\overline{u}, \delta)$. Consider the IVP

$$\begin{cases} \dfrac{dv}{dt} = f(v) \\ v(t^*) = u^* \end{cases}$$

Where $t^* = t_m$ and $u^* = u(t_m)$. By what we showed above, this IVP has a unique solution $v(t)$ for $t = [t^* - r, t^* + r]$.
STEP 3: By uniqueness (since f is locally Lipschitz), we can combine these solutions together, since we can stop $u(t)$ at (t^*, u^*), and $v(t)$ starts at (t^*, u^*). Thus we have the following solution to the original IVP:

$$w(t) = \begin{cases} u(t) & T \in [0, t^*] \\ v(t) & T \in [t^*, t^* + r]. \end{cases}$$

Since $t^* + r > T$, this solution exists on a larger interval than $[0, T)$ $\Rightarrow\Leftarrow$ $\square$

The existence result from the Picard-Lindelöf theorem is only a *local* existence result. If we have a linear system, however, we have a global existence result:

Global Existence for Linear Systems:

Consider the system

$$\begin{cases} \dfrac{du}{dt} = A(t)u \\ u(0) = u_0, \end{cases}$$

where $u \in \mathbb{R}^n$ and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous. Then there exists a unique solution $u(t)$ which exists for all $t \in \mathbb{R}$.

Proof:

STEP 1: Write the ODE as $\dot{u} = f(u,t)$ with $f(u,t) = A(t)u$. Since $f(u,t)$ is Lipschitz in u on every bounded interval $[-T,T]$ the IVP has a unique solution $u(t)$ for t in an interval containing 0.

We wish to show that $u(t)$ exists for all $t \in \mathbb{R}$. Suppose this solution exists on $[0,t_0)$, but not on any larger interval $[0,t_0+\epsilon)$. Then $u(t)$ must blow up as it approaches t_0.

STEP 2: Integrate the ODE

$$u(t) = u_0 + \int_0^t A(\tau)u(\tau)d\tau \qquad\qquad t \in [0,t_0)$$

$$\text{Hence } |u(t)| \le |u_0| + \int_0^t \|A(\tau)\|\, |u(\tau)|d\tau \qquad t \in [0,t_0)$$

Here $\|A(\tau)\|$ is the operator (matrix) norm of $A(t)$

STEP 3: This satisfies the hypotheses of Gronwall's Inequality. Thus, for $t \in [0, t_0)$

$$|u(t)| \leq |u_0| \exp\left(\int_s^t \|A(\tau)\| d\tau \right) \leq |u_0| \exp\left(\int_s^{t_0} \|A(\tau)\| d\tau \right) \leq C|u_0|$$

Hence $u(t)$ cannot blow up as $t \to t_0$, it must $u(t)$ must exist for all $t \geq 0$. A similar argument shows that $u(t)$ must exist for all $t \leq 0$.