LECTURE: SECOND-ORDER ODE

Today: How to solve second-order ODE?

1. Model Problem

Video: Differential Equations the COOL way

Example 1:

$$y'' - 5y' + 6y = 0$$

Note: In the Appendix, you can find another way of doing this.

We'll solve this by "factoring out" the differential equation

Definition: (Differential Operators)

$$Dy = y' \quad \text{and} \quad D^2 y = y''$$

STEP 1: Notice you can write the ODE in terms of D:

$$\begin{aligned} y'' - 5y' + 6y &= 0 \\ D^2 y - 5Dy + 6y &= 0 \\ (D^2 - 5D + 6)y &= 0 \end{aligned}$$

Trick: Factor this out as $(D-2)(D-3)y = 0$

STEP 2: In particular, if you let $z = (D-3)y$ then the ODE becomes

$$\begin{aligned}(D-2)\underbrace{(D-3)y}_{z} =& 0\\ (D-2)z =& 0\\ Dz - 2z =& 0\\ z' - 2z =& 0\\ z' =& 2z\\ z =& Ae^{2t}\end{aligned}$$

STEP 3: Now solve for y

$$\begin{aligned}z =& Ae^{2t}\\ (D-3)y =& Ae^{2t}\\ y' - 3y =& Ae^{2t}\end{aligned}$$

Use the integrating factor e^{-3t}

$$\begin{aligned}e^{-3t}y' - 3e^{-3t}y =& Ae^{2t}e^{-3t}\\ \left(e^{-3t}y\right)' =& Ae^{-t}\\ e^{-3t}y =& \int Ae^{-t}dt = -Ae^{-t} + B\\ y =& e^{3t}\left(-Ae^{-t} + B\right)\\ y =& \underbrace{-A}_{A}e^{2t} + Be^{3t}\\ y =& Ae^{2t} + Be^{3t}\end{aligned}$$

Note: Intuitively this makes sense: for *first*-order equations we had *one* constant C, so for *second*-order equations, we have *two* constants, A and B.

Faster Way: Notice the exponents 2 and 3 come from factoring

$$D^2 - 5D + 6 = (D-2)(D-3)$$

Definition:

The **auxiliary equation** of $y'' - 5y' + 6y = 0$ is $r^2 - 5r + 6 = 0$

Since $r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2$ or $r = 3$ we get

$$y = Ae^{2t} + Be^{3t}$$

But the real reason this works is because of the factoring method above.

2. Examples

Upshot: To solve second-order ODE, you just need to find the roots of the auxiliary equation.

Example 2:

$$y'' + 5y' + 4y = 0$$

Aux: $r^2 + 5r + 4 = 0 \Rightarrow (r+1)(r+4) = 0 \Rightarrow r = -1$ or $r = -4$

$$y = Ae^{-t} + Be^{-4t}$$

Example 3:

$$\begin{cases} y'' + y' - 6y = 0 \\ y(0) = 2 \\ y'(0) = -1 \end{cases}$$

Aux: $r^2 + r - 6 = 0 \Rightarrow (r+3)(r-2) = 0 \Rightarrow r = -3$ or $r = 2$

$$y = Ae^{-3t} + Be^{2t}$$

$$y(0) = 2 \Rightarrow Ae^{-3(0)} + Be^{2(0)} = 2 \Rightarrow A + B = 2 \Rightarrow B = 2 - A$$

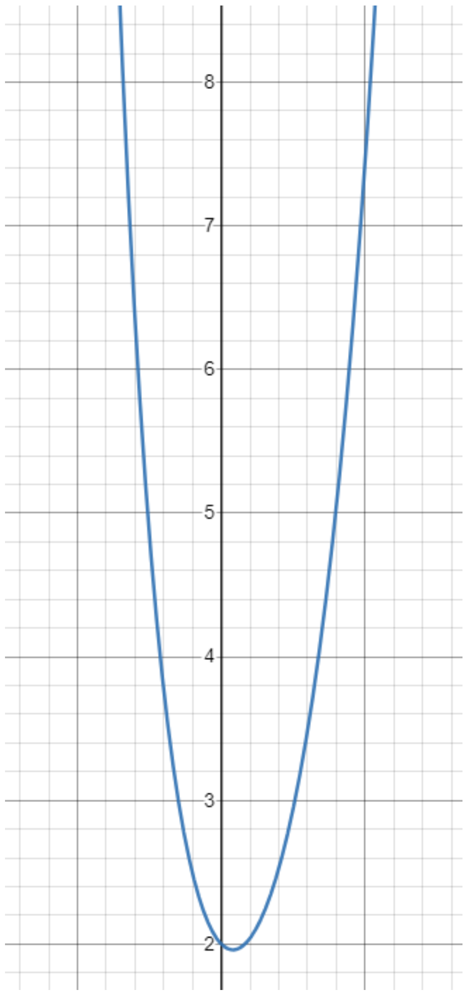
$$y = Ae^{-3t} + (2-A)e^{2t}$$

$$y'(t) = A\left(-3e^{3t}\right) + (2-A)2e^{2t}$$

$$\begin{aligned} y'(0) &= -3Ae^0 + 2(2-A)e^0 \\ &= -3A + 4 - 2A \\ &= -5A + 4 \\ &= -1 \end{aligned}$$

Hence $-5A = -5 \Rightarrow A = 1$

$$y = Ae^{-3t} + (2-A)e^{2t} = e^{-3t} + e^{2t}$$



(It's **not** a parabola, but rather a hyperbolic function like cosh)

Example 4: (More practice)

When does y in the previous example attain its minimum?

From Calculus 1, all you need to do here is to set $y' = 0$

$$\begin{aligned}
y' &= 0 \\
\left(e^{-3t} + e^{2t}\right)' &= 0 \\
-3e^{-3t} + 2e^{2t} &= 0 \\
2e^{2t} &= 3e^{-3t} \\
\frac{e^{2t}}{e^{-3t}} &= \frac{3}{2} \\
e^{5t} &= \frac{3}{2} \\
5t &= \ln\left(\frac{3}{2}\right) \\
t &= \left(\frac{1}{5}\right)\ln\left(\frac{3}{2}\right) \\
t &\approx 0.081
\end{aligned}$$

Sometimes you have to use the quadratic formula:

Example 5: (More practice)

$$4y'' = 8y' - 3y$$

Same as $4y'' - 8y' + 3y = 0$

$$
\begin{aligned}
4r^2 - 8r + 3 &= 0 \\
r &= \frac{8 \pm \sqrt{(-8)^2 - 4(4)(3)}}{2(4)} \\
&= \frac{8 \pm \sqrt{16}}{8} \\
&= \frac{8 \pm 4}{8} \\
&= \frac{4}{8} \text{ or } \frac{12}{8} \\
r &= \frac{1}{2} \text{ or } \frac{3}{2}
\end{aligned}
$$

$$y = Ae^{\frac{t}{2}} + Be^{\frac{3t}{2}}$$

The cool thing is that **everything** we learn about second order equations generalizes easily to higher-order ones as well:

Example 6:

$$y''' - 6y'' + 11y' - 6y = 0$$

Aux: $r^3 - 6r^2 + 11r - 6 = 0$

You are **not** responsible for knowing how to factor out **cubic** polynomials, but suppose someone tells you that this factors as

$$(r-1)(r-2)(r-3) = 0 \Rightarrow r = 1 \text{ or } r = 2 \text{ or } r = 3$$

Then the general solution is

$$y = Ae^{t} + Be^{2t} + Ce^{3t}$$

3. Repeated Roots

Video: Cool Repeated Roots

What if our auxiliary equation only has one root?

Example 7:

$$y'' - 4y' + 4y = 0$$

Aux: $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ Repeated Twice

The only difference is that, instead of writing $Ae^{2t} + Be^{2t}$ which is redundant, you add an extra t to the second term:

Fact:

$$y = Ae^{2t} + Bte^{2t}$$

Why? Let's use the factoring method again!

STEP 1:

$$\begin{aligned} (D^2 - 4D + 4)y &= 0 \\ (D-2)\underbrace{(D-2)y}_{z} &= 0 \\ z' - 2z &= 0 \\ z &= Ae^{2t} \end{aligned}$$

STEP 2:

$$
\begin{aligned}
(D-2)y =& z \\
y' - 2y =& Ae^{2t} \\
e^{-2t}y - 2e^{-2t}y =& Ae^{2t}e^{-2t} \\
\left(e^{-2t}y\right)' =& A \\
e^{-2t}y =& \int A dt = At + B \qquad \text{(here is where the } t \text{ comes from)} \\
y =& Ate^{2t} + Be^{2t}
\end{aligned}
$$

Note: Another way to show this (which was on the homework) is

$$
\begin{aligned}
\left(e^{-2t}y\right)'' =& \left(-2e^{-2t}y + e^{-2t}y'\right)' \\
=& (-2)(-2)e^{-2t}y - 2e^{-2t}y' - 2e^{-2t}y' + e^{-2t}y'' \\
=& 4e^{-2t}y - 4e^{-2t}y' + e^{-2t}y'' \\
=& e^{-2t}\underbrace{\left(y'' - 4y' + 4y\right)}_{0} \\
=& 0
\end{aligned}
$$

$$
\begin{aligned}
&\left(e^{-2t}y\right)'' = 0 \\
\Rightarrow &\left(e^{-2t}y\right)' = B \\
\Rightarrow &e^{-2t}y = Bt + A \\
\Rightarrow &y = e^{2t}\left(A + Bt\right) \\
\Rightarrow &y = Ae^{2t} + Bte^{2t}
\end{aligned}
$$

Which is the same solution as above!

Example 8:

$$\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 1 \\ y'(0) = -3 \end{cases}$$

Aux: $r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r = 3$ Repeated

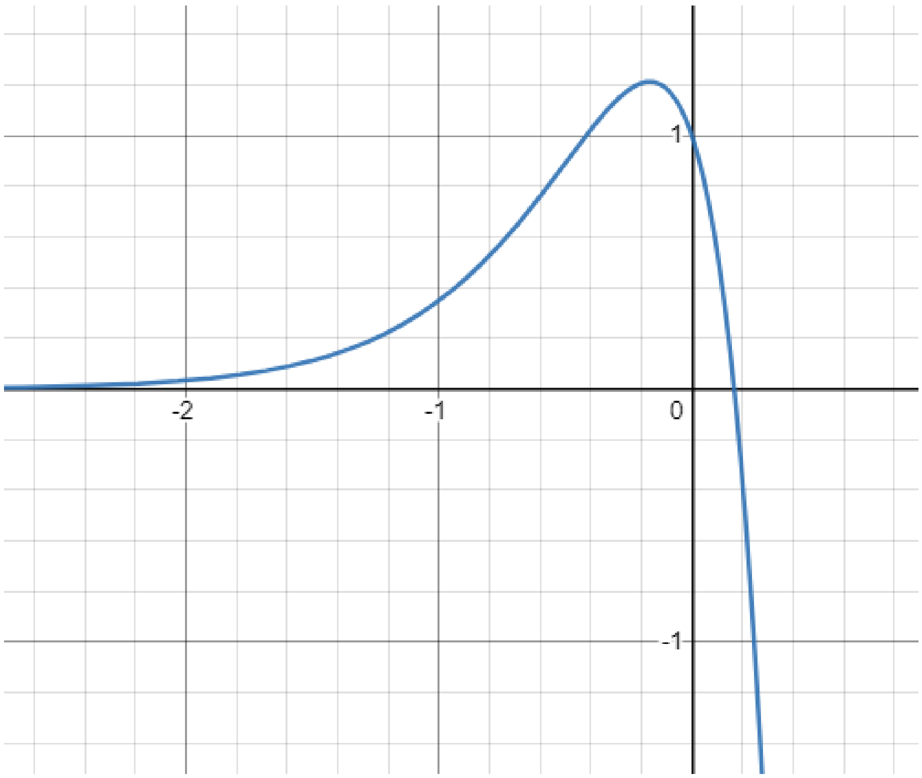
$$y = Ae^{3t} + Bte^{3t}$$

$$y(0) = 1 \Rightarrow Ae^0 + B(0)e^0 = 1 \Rightarrow A = 1$$

$$y' = \left(e^{3t} + Bte^{3t}\right)' = 3e^{3t} + Be^{3t} + Bt(3e^{3t})$$

$$\begin{aligned} y'(0) &= -3 \\ 3e^0 + Be^0 + B(0)(3e^0) &= -3 \\ 3 + B &= -3 \\ B &= -3 - 3 \\ B &= -6 \end{aligned}$$

$$y = e^{3t} - 6te^{3t}$$



Based on the graph above, it looks like the solutions has a maximum slightly before $t = 0$. Let's find it!

Example 9: (extra practice)

Where does y as above attain a max?

We just need to set $y' = 0$

$$\begin{aligned} y' &= \left(e^{3t} - 6te^{3t}\right)' \\ &= 3e^{3t} - 6e^{3t} - 6t\left(3e^{3t}\right) \\ &= -3e^{3t} - 18te^{3t} \\ &= (-3 - 18t)\, e^{3t} \\ &= 0 \end{aligned}$$

$$-3 - 18t = 0 \Rightarrow t = \frac{-3}{18} = -\frac{1}{6} \approx -0.1667$$

4. Appendix: Other Method

Here is another way of solving second-order ODE, which is easier to use, but less intuitive:

Example 10:

Solve $y'' - 5y' + 6y = 0$

We'll do this by analogy with first order equations:

Recall: (Basic ODE)

$$y' = ry \Rightarrow y = Ce^{rt}$$

Idea: Exponential functions are so useful, let's try out the same guess.

Plug in e^{rt} in $y'' - 5y' + 6y = 0$:

$$\begin{aligned}\left(e^{rt}\right)'' - 5\left(e^{rt}\right)' + 6e^{rt} &= 0\\ r^2e^{rt} - 5re^{rt} + 6e^{rt} &= 0\\ \left(r^2 - 5r + 6\right)\cancel{e^{rt}} &= 0\\ r^2 - 5r + 6 &= 0\end{aligned}$$

$$r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2, 3$$

- $r = 2$ and $r = 3$ tells us that e^{2t} and e^{3t} are solutions
- A constant times a solution is still a solution, so Ae^{2t} and Be^{3t} are solutions

- The sum of two solutions is still a solution, so $Ae^{2t} + Be^{3t}$ is a solution. This gives us *one* solution of the ODE.
- Using $y(0)$ and $y'(0)$ we can (in theory) solve for A and B
- Uniqueness says this is the only solution, so $y = Ae^{2t} + Be^{3t}$

Fact:

The general solution is $y = Ae^{2t} + Be^{3t}$ where A and B are constants.

Note: Intuitively this makes sense: for *first*-order equations we had *one* constant C, so for *second*-order equations, we have *two* constants, A and B.

Note: In theory, it's possible to get redundant solutions, like e^{2t} and e^{2t} There is a tool called the **Wronskian** that checks that if the solutions are *linearly independent* (not redundant) or not.