APMA 1650 – Homework 2

1. Suppose you flip a fair coin 50 times. Answer the following questions. You may leave the answers in terms of binomial coefficients and exponents, i.e. you don't have to expand out all the things.

   (a) What is the size of the sample space for this experiment, i.e. how many outcomes are possible?

   Since there are two possible outcomes for each coin flip (H or T), and there are 50 flips, by the Basic Principle of Counting the sample space contains 2^{50} possible outcomes.

   (b) What is the probability that you flip exactly 10 heads?

   An outcome can be represented as a string of 50 Hs and Ts. Exactly 10 heads means a string composed of 10 Hs and 40 Ts. Since there are $\binom{50}{10}$ such strings, the probability is:

   $$\frac{\binom{50}{10}}{2^{50}}$$

   (c) What is the probability that you flip at least 10 heads? At least 10 heads means not flipping $0, 1, 2, \ldots, 9$ heads. Thus we have the probability of at least 10 heads in a row is:

   $$1 - \frac{\binom{50}{0}}{2^{50}} - \frac{\binom{50}{1}}{2^{50}} - \cdots - \frac{\binom{50}{9}}{2^{50}}$$

   You could also sum up the probabilties of $10, 11, 12, \ldots, 50$ heads.

   (d) What is the probability that you never flip two heads in a row or two tails in a row? The only way to do this is to alternate heads and tails, i.e. to flip HTHT...HT or THTH...TH. Thus there are only two ways to do this, so the probability is:

   $$\frac{2}{2^{50}} = \frac{1}{2^{49}}$$

2. You enlist a friend from materials science to construct a very special unfair six-sided die. The die looks like a standard die, i.e. it it cubical and has the numbers 1, 2, 3, 4, 5, and 6 on its faces. On this die, the probability of rolling any number is directly proportional to that number. For example, you are twice as likely to roll a 6 as a 3.

   (a) What is the probability of rolling each of the six numbers?
   Let p be the the probabiltiy of rolling a 1. Then the probability of rolling a 2 is $2p$, 3 is $3p$, etc. In other words, the probability of rolling k is kp, $k = 1, 2, 3, 4, 5, 6$. Since the probabilities must sum to 1:

   $$1 = p + 2p + 3p + 4p + 5p + 6p = 21p$$

   Thus $p = 1/7$. We then have probabilties:

Die roll	probability
1	1/21
2	2/21
3	3/21
4	4/21
5	5/21
6	6/21

(b) What is the probabiltiy of rolling an odd number?
Adding up the probabilities for 1, 3, and 5, we get 9/21.

3. Consider the grid of points shown below. You start at (0, 0) and take one step either up or to the right with each move. You keep moving until you reach (10, 10). You can never leave the grid. For example, if you reach the point (4, 10), you can only move to the right from that point onwards. Assume that each possible path is equally likely. Answer the following questions. You may leave the answers in terms of binomial coefficients and exponents.

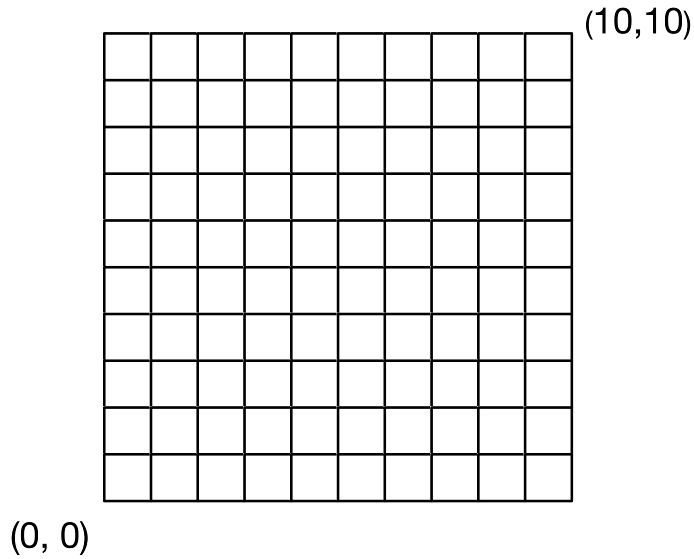
   (a) What is the size of the sample space, i.e. how many possible paths are there from (0, 0) to (10, 10)?

   To get from (0, 0) to (10, 10) we must take 10 steps right and 10 steps up. These may be done in any order. If we think of a path as string of 20 Us or Rs, where we have exaclty 10 Us and 10 Rs, the number of such strings (thus the number of possible paths) is $\binom{20}{10}$.

   (b) What is the probability that a path passes through (5, 5)?

   A path that goes through (5,5) must go from (0, 0) to (5, 5) then from (5, 5) to (10, 10). Using the logic above, there are $\binom{10}{5}$ paths from (0, 0) to (5, 5) and $\binom{10}{5}$ paths from (5, 5) to (10, 10). Multiplying these together, there are $\binom{10}{5}^2$ paths from (0, 0) to (10, 10) which pass through (5, 5). Thus the probability of this occurring is:

$$\frac{\binom{10}{5}^2}{\binom{20}{10}}$$



4. Powerball is an American lottery game offered by 44 states (including Rhode Island). To play the game, you select 5 distinct numbers from a set of 69 white balls (numbered 1 - 69) and one number from a set of 26 red Powerballs (numbered 1 - 26). In each drawing, five white balls and one red Powerball are selected[1]. The order of the white balls does not matter. (In the official Powerball drawing, the white balls are listed in ascending order.)

   (a) You win the jackpot if you match all 5 white balls and the Powerball. What is the probability that you win the jackpot?

   First note that there are $\binom{69}{5}$ possible draws for the 5 white balls (since order does not matter). To win the jackpot, you need to get 1 out of these $\binom{69}{5}$ possible draws for the 5 white balls and 1 out of the 26 possible draws for the powerball. The probability is:

$$\frac{1}{\binom{69}{5}}\frac{1}{26}, \text{ approximately 1 in 292,201,338}$$

   (b) If you match all 5 white balls but do not match the Powerball, you win \$1,000,000. What is the probability that this occurs?

   This is the same as the above except we don't want to match the Powerball. Since there are 25/26 ways to not match the Powerball, this is:

$$\frac{1}{\binom{69}{5}}\frac{25}{26}, \text{ approximately 1 in 11,688,054}$$

[1]If you Google "Powerball", today's numbers are displayed at the top of the search results. In asking this question, the instructor is not condoning playing Powerball.

(c) If you match the Powerball but do not match *any* of the white balls, you win \$4. What is the probability that this occurs?

To not match any of the white balls, your 5 balls must all be chosen from the 64 balls which are not winners. There are $\binom{64}{5}$ ways to choose 5 balls from the 64 non-winning balls. Since there is a 1/26 probability of matching the Powerball, the probability of this event is:

$$\frac{\binom{64}{5}}{\binom{69}{5}}\frac{1}{26}, \text{ approximately 1 in 38}$$

(d) If you match *exactly* 3 white balls (so you don't match the other two balls) and the Powerball, you win \$100. What is the probability that this occurs?

To match exactly 3 white balls, you choose 3 balls from the 5 winning balls ($\binom{5}{3}$ ways to do this) and you choose 2 bals from the 64 non-winning balls ($\binom{64}{2}$ ways to do this). Since there is a 1/26 probability of matching the Powerball, the probability of this event is:

$$\frac{\binom{5}{3}\binom{64}{2}}{\binom{69}{5}}\frac{1}{26}, \text{ approximately 1 in 14,494}$$

5. How many distinct arrangements are there of the word HULLABALLOO?

$S = \{\text{AABHLLLLOOU}\}$

There are 11 letters in HULLABALLOO, if all of the letters were distinct then there would be 11! ways to arrange the letters, since there are some repeats you account for this by dividing by the number of ways to arrange the repeated letters, for example there are 4 L's so you divide 11! by 4!, thus the answer is:

$$\frac{11!}{4! * 2! * 2!} = \frac{11!}{96} = 415,800$$

6. Poker. Total number of ways to draw a 5-card hand from 52 cards:

$$\binom{52}{5} = 2598960$$

ways.

(a) Royal Flush: 10 J Q K A of the same suit:

There are 4 suits in a deck (Hearts, Diamonds, Clubs, Spades), so there are

$$4$$

possible royal flushes.

Probability: $\frac{4}{\binom{52}{5}} \approx 1.54 \times 10^{-6}$

(b) Straight flush (excluding royal flush): Any consecutive order of cards of the same suit, like 45678.

There are 9 possible non-royal straight flushes in each suit (A 2 3 4 5, 2 3 4 5 6, ..., 9 10 J Q K). So, there are

$$9 \times 4 = 36$$

possible non-royal straight flushes.

Probability: $\frac{36}{\binom{52}{5}} \approx 0.0000139$

(c) Four of a kind, like 3333A

For the card that we have four of, there are 13 choices. For the fifth card, there are 48 choices (since it can't be the card that we have four of). Thus, there are

$$13 \times 48 = 624$$

possible four of a kinds.

Probability: $\frac{624}{\binom{52}{5}} \approx 0.000240$

(d) Flush (excluding royal and straight flush): Five cards of the same suit

There are $\binom{13}{5}$ ways to choose 5 cards from one suit. There are 4 suits, so there are $4 \times \binom{13}{5}$ flushes. Subtracting the straight flushes and royal flushes yields

$$4\binom{13}{5} - 40 = 5108$$

possible flushes.

Probability: $\frac{5108}{\binom{52}{5}} \approx 0.00197$

(e) Straight (excluding royal flush and straight flush): Any consecutive order of cards, not necessarily of the same suit, like 78910J

There are 10 possible suit-independent straights (A 2 3 4 5, 2 3 4 5 6, ..., 10 J Q K A). Each straight can be in any of 4^5 suit combinations. However, we have to subtract the suit combinations that give a straight flush or a royal flush. Hence, there are

$$10(4^5 - 4) = 10200$$

possible straights.

Probability: $\frac{10200}{\binom{52}{5}} \approx 0.00392$

(f) Three of a kind, like 444JK

There are 13 choices for the rank that we have three of. For the two singletons, there are $\binom{12}{2}$ choices. For suits, there are $\binom{4}{3}$ ways to choose the suits for the three of a kind, and 4 ways to choose the suit for each of the two singletons. Thus, there are

$$13 \times \binom{12}{2} \times \binom{4}{3} \times 4 \times 4 = 54912$$

possible three of a kinds.

Probability $= \frac{54912}{\binom{52}{5}} \approx 0.0211$

(g) Two pairs, like 33558

There are $\binom{13}{2}$ ways to choose the two ranks for the pairs. For the singleton, there are 11 choices. For the suits, there are $\binom{4}{2}$ ways to choose the suits for each pair and 4 ways to choose the suit for the singleton. Thus, there are

$$\binom{13}{2} \times 11 \times \binom{4}{2} \times \binom{4}{2} \times 4 = 123552$$

possible two pairs.

Probability $= \frac{123552}{\binom{52}{5}} \approx 0.0475$

(h) One pair, like 224JK

There are 13 choices for the rank of the pair. For the three singletons, there are $\binom{12}{3}$ choices. For the suits, there are $\binom{4}{2}$ ways to choose the suits for the pair, and $4 \times 4 \times 4$ ways to choose the suits for the three singletons. Thus, there are

$$13 \times \binom{12}{3} \times \binom{4}{2} \times 4 \times 4 \times 4 = 1098240$$

possible one pairs.

Probability $= \frac{1098240}{\binom{52}{5}} \approx 0.423$