LECTURE: WHAT IS A PROBABILITY?

1. Introduction

The probability of an event measures the chance of it happening

Examples: Here are some fun probabilities[1]

(1) Probability of winning an oscar: 1 in $11,500$

(2) Probability of being injured by a toilet: 1 in $10,000$

(3) Probability of being attacked by a shark: 1 in $3,500,000$

(4) Probability of being struck by lightning: 1 in $115,000$

(5) Probability of winning the Powerball jackpot: 1 in $292,000,000$

Why Probability: Some people believe probabilities don't exist: either an event happens or it doesn't. But you can't deny that a probability influences at least your thinking about that event.

Example: The weather report says "There's a 90 % chance of raining tomorrow," chances are you will be bringing an umbrella with you, or at least thinking about bringing it.

Applications: Here are some cool real-life uses of probability:

[1] Source

(1) **Insurance:** Used by insurance companies to determine the price of your policy.

(2) **Finance:** Generally you want to invest in stocks that have a lower chance of failure

(3) **Biology:** Disease spread, chances that a surgery will be successful, life expectancy etc.

(4) **Warranty:** Manufacturers of auto parts calculate the probability of a part failing to determine its warranty

(5) **Image processing:** If you have a picture that's destroyed or just partially colored, you can use probability to reconstruct the image! For instance, if a pixel is red, chances are the neighboring pixels are a shade of red as well

2. Classical vs Empirical Approach

There are two main ways of calculating probabilities.

Example: Suppose you roll a 6-sided die, what is the probability of rolling a 3?

Classical approach: Since there are 6 sides to a die and each side is equally likely to happen, then the probability is $\frac{1}{6}$

The advantage of this approach is that we can calculate this in our heads, but the limitation is that we need a symmetry argument for this to be effective. So it works for coin flips and die rolls, but not for more complicated scenarios.

Empirical approach: Imaging rolling a standard six-sided die $10,000$ times. The empirical probability of rolling a 3 is the ratio of the number of times you actually get a 3 divided by the total number of rolls $10,000$

In general, we have:

$$\text{Empirical Probability } = \frac{\text{Number of times the event occurs}}{\text{Total number of trials}}$$

Intuitively, as we perform more and more dice rolls, the empirical probability of rolling a 3 should approach some fixed number. That number is what we call the true probability of rolling a 3.

For example, if you use the following simulator then the number of times you roll a 3 out of $10,000$ is 1687 so the probability is close to $\frac{1687}{10{,}000}$ If you increase the number of rolls, you get a quantity that is close to $\frac{1}{6}$ so the empirical probability in this case is $\frac{1}{6}$

Fact: (Weak Law of Large Numbers)

As the number of rolls n approaches ∞, the empirical probability will approach the classical probability $\frac{1}{6}$

The empirical approach has a critical advantage over the classical approach in that we do not require symmetry to compute our probabilities and that we can just use simulations. You can also use it to determine if a die is crooked. The disadvantage is that you cannot simulate every experiment in the world. For example, you cannot think of the chance of rain tomorrow as the limit of a sequence of independent experiments, unless we are in the movie Groundhog day ☺

3. Sample Spaces

Definition: A **set** is a collection of distinct objects

Definition: A **sample space** S is the set of all outcomes of a particular experiment

Think of S as being our universe, all the events that could possibly happen.

Examples:

(1) Single coin flip: $S = \{H, T\}$

(2) Roll a six-sided standard die: $S = \{1, 2, 3, 4, 5, 6\}$

(3) Roll two six-sided standard die.

Roll 1 \ Roll 2	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
2	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
3	$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
4	$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
5	$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
6	$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

(4) Number of free throw attempts it takes for me to make a single basket: $S = \mathbb{N} = \{1, 2, 3, \cdots\}$

(5) Number of minutes late my bus arrives $S = [0, \infty)$

Note that the first three sample spaces contain only a finite number of elements (2, 6, and 36 elements, respectively). These are called **finite**

sample spaces

Note: The main difference between $\mathbb{N}$ and $[0, \infty)$ is that the former is **countable** whereas the latter is **uncountable**. Countable intuitively means that you can count all the elements in the set with your hand, provided you have enough time. We cannot do this with $[0, \infty)$: There are way too many elements in that set to list them, even if we had an infinite amount of time. A proof of this fact is left for another course and can be found here.

Definition: A sample space which is either finite or countably infinite is called **discrete**.

It's basically everything except for uncountable spaces.

4. Events and Subsets

Definition: An **event** is a subset of a sample space.

Notation: We usually denote events by A and B, as opposed to the sample space S

Definition: A is a subset of B if for all x in A, we also have x is in B. We write $A \subseteq B$

The **empty set** denoted by $\emptyset$ is the set containing no elements, and is a subset of every set

Example: Consider again rolling a die and let our sample space be $S = \{1, 2, 3, 4, 5, 6\}$

Then the following are examples of events:

(1) An even number is rolled: $A = \{2, 4, 6\}$

(2) The roll is ≤ 3: $B = \{1, 2, 3\}$

(3) 3 is rolled: $C = \{3\}$

Notice C consists of a single element in the sample space. Such an event is called a **simple event** and cannot be decomposed.

The events A and B are each composed of three simple events.

Example: Consider the sample space of rolling two dice. Let E be the event that the sum of the two dice is 7. We can represent this graphically:

Roll 1 \ Roll 2	1	2	3	4	5	6
1	$(1, 1)$	$(1, 2)$	$(1, 3)$	$(1, 4)$	$(1, 5)$	$(1, 6)$
2	$(2, 1)$	$(2, 2)$	$(2, 3)$	$(2, 4)$	$(2, 5)$	$(2, 6)$
3	$(3, 1)$	$(3, 2)$	$(3, 3)$	$(3, 4)$	$(3, 5)$	$(3, 6)$
4	$(4, 1)$	$(4, 2)$	$(4, 3)$	$(4, 4)$	$(4, 5)$	$(4, 6)$
5	$(5, 1)$	$(5, 2)$	$(5, 3)$	$(5, 4)$	$(5, 5)$	$(5, 6)$
6	$(6, 1)$	$(6, 2)$	$(6, 3)$	$(6, 4)$	$(6, 5)$	$(6, 6)$

Mathematically you can write this as

$$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$