LECTURE: BINOMIAL DISTRIBUTION

1. Bernoulli Trials

One of the simplest probability models is that of **Bernoulli trials**. It's just a generalization of coin tossing

Definition:

A **Bernoulli trial** is a sequence of experiments with the following properties:

(1) Each trial has exactly two possible outcomes, designated *success* and *failure*.

(2) The trials are independent

(3) For each trial, the probability of success is p and the probability of failure is $1 - p$ where $0 \leq p \leq 1$

Examples:

(1) Flipping a coin, where success = head and failure = tails (or vice-versa)/ If it is a fair coin, then $p = 1/2$

(2) Rolling two dice, where success = roll of doubles (like 33) This is how you get out of jail in Monopoly.

(3) Playing a slot machine in Las Vegas.

Each individual trial is modeled with a Bernoulli random variable

Definition:

A **Bernoulli random variable** Y with parameter p is a random variable such that

$$Y = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

We write $Y \sim \text{ Ber } (p)$

It models a single Bernoulli trial, where 1 indicates success and 0 indicates failure. Its pmf is as follows:

y	$p(y)$
0	$1-p$
1	p

Example 1:

Suppose $Y \sim \text{Ber}(p)$. What is the mean and variance of Y?

$$E(Y) = \sum_{\text{all } y} y\, p(y) = 0(1-p) + 1(p) = p$$

Recall: Magic Variance Formula

$$\text{Var}(Y) = \left(E(Y^2)\right) - [E(Y)]^2$$

$$E(Y^2) = \sum_{\text{all } y} y^2\, p(y) = 0^2(1-p) + 1^2(p) = p$$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2 = p - p^2 = p(1-p)$$

Facts:

If $Y \sim$ Ber (p) then $E(Y) = p$ and $\text{Var}(Y) = p(1-p)$

Note: Notice $f(p) = p(1-p)$ is a parabola with a maximum at $p = \frac{1}{2}$

Bernoulli trials are useful in many situations. Here are two questions we might want to ask about them:

(1) How many successes do we have out of a *fixed* number of trials?

(2) How many trials does it take to get our first success?

The first question is answered by the binomial distribution, and the second by the geometric distribution.

2. Binomial Distribution

It models the number of successes in a number of Bernoulli trials.

Example 2:

Suppose you play a slot machine 10 times, where the probability of winning one play is p. What is the probability that you win exactly 2 times?

This is an example of a sequence of Bernoulli trial.

Notice that any sequence of 10 plays can be represented in the form `WLWWLWLWLL` where `W` is Win and `L` is lose.

We are looking for sequences with exactly two `W`, such as `LLWLLWLLLL`

How many such sequences are there?

This is like asking "What is the probability of getting exactly 2 heads in 10 coin tosses?" Which is $\binom{10}{2}$

On the other hand, what is the probability of getting such a sequence, like `LLWLLWLLLL`?

By independence, we have

$$P(\texttt{LLWLLWLLLL}) = (1-p)(1-p)p(1-p)(1-p)p(1-p)(1-p)(1-p)(1-p) = p^2(1-p)^8$$

So in the end, we get

$$P(2 \text{ wins out of } 10 \text{ trials}) = \binom{10}{2} p^2(1-p)^8$$

More generally, if you have n trials and want y successes, think y heads in n coin tosses, the number of such sequences is $\binom{n}{y}$ and the probability of each sequence is $p^y(1-p)^{n-y}$ therefore

$$P(y \text{ successes out of } n \text{ trials}) = \binom{n}{y} p^y(1-p)^{n-y}$$

Definition:

A discrete random variable Y has a **binomial distribution** with n trials and probability of success p if

$$p(y) = \binom{n}{y} p^y (1-p)^{n-y} \qquad\qquad y = 0, 1, \ldots, n$$

We write $Y \sim$ Binom (n, p)

This models the number of successes out of n Bernoulli trials, where the probability of a single success is p.

Let's check that Y is a well-defined discrete random variable, i.e. the probabilities of all its possible outputs sum to 1.

Recall: Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Using the binomial theorem, we see that

$$\sum_{y=0}^{n} p(y) = \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} = [p + (1-p)]^n = 1^n = 1$$

Example 3:

Let $Y \sim \text{Binom}(n, p)$. Calculate $E(Y)$ and $\text{Var}(Y)$

We could do this directly, but because of the symmetry, let's use the method of indicators!

For each trial $i = 1, 2, \ldots, n$ let:

$$Y_i = \begin{cases} 1 & \text{if trial } i \text{ is a success} \\ 0 & \text{if trial } i \text{ is a failure} \end{cases}$$

Notice that each $Y_i \sim \text{Ber}(p)$ and moreover

$$\text{Then } Y = Y_1 + Y_2 + \cdots + Y_n$$

$$\begin{aligned} \text{And } E(Y) &= E(Y_1 + Y_2 + \cdots + Y_n) \\ &= E(Y_1) + E(Y_2) + \cdots + E(Y_n) \\ &= p + p + \cdots + p = np \end{aligned}$$

We can do the same thing for the variance, since the Y_i are "independent" (since Bernoulli trials are independent), so

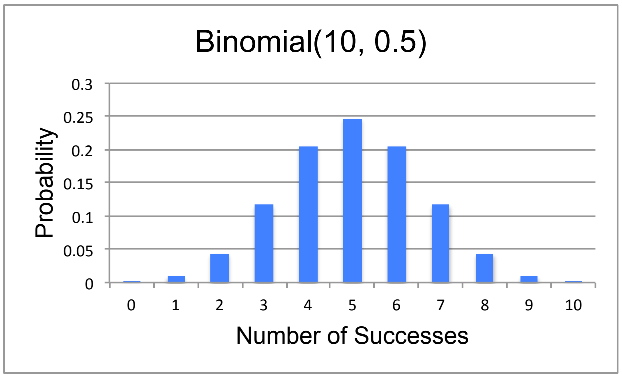
$$\begin{aligned} \text{Var}(Y) &= \text{Var}(Y_1 + Y_2 + \cdots + Y_n) \\ &= \text{Var}(Y_1) + \text{Var}(Y_2) + \cdots + \text{Var}(Y_n) \\ &= p(1-p) + p(1-p) + \cdots + p(1-p) = np(1-p) \end{aligned}$$

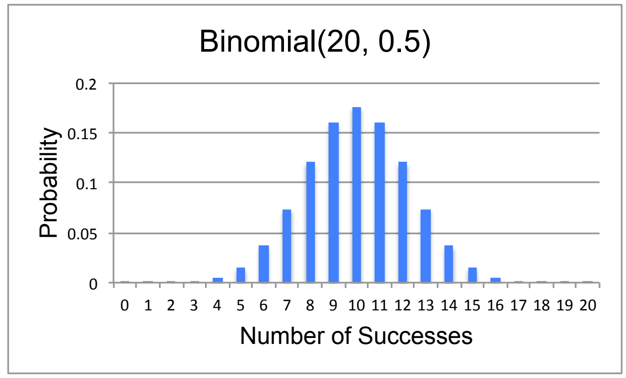
Fact:

If $Y \sim$ Binom (n, p), then $E(Y) = np$ and $\text{Var}(Y) = np(1-p)$

Histograms: Let's look at histograms of the binomial distribution for a few choices of parameters.

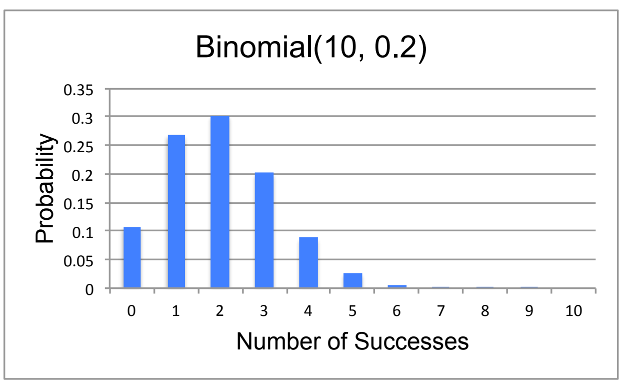
First here are histograms for $p = 1/2$ (fair coin):

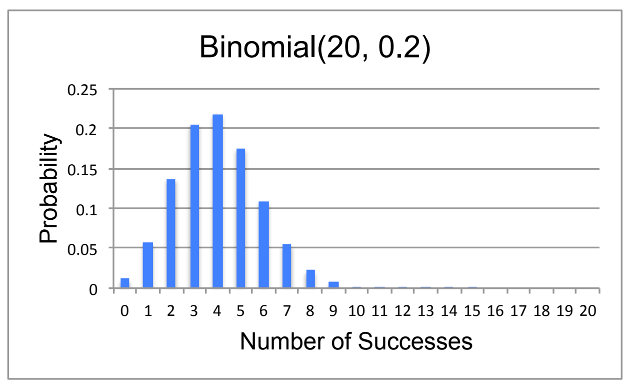




These distributions are perfectly symmetric around the mean, which is expected for the case where $p = 1/2$. Note that as n increases, these look more and more like "bell curves". For large enough n, we will be able to approximate the (discrete) binomial distribution by the (continuous) normal distribution.

The histograms look a bit different for p significantly different from $1/2$. These histograms are for $p = 0.2$.





These are not symmetric. Since $p < 1/2$, the distributions are skewed to the left, which is what we expect since failure is more likely that success. Although not to the same extent as the case where $p = 1/2$, these also start to look like bell curves as n increases. We will also be able to approximate these by normal distributions for large n, but the farther p is from $1/2$, the larger n will need to be for this approximation to be reasonable.

Example 4:

When a certain variety of pea plant is cross-fertilized, the offspring have white flowers $1/4$ of the time and purple flowers $3/4$ of the time

(a) You cross-fertilize 20 of these pea plants. What is the the probability that 5 of them are white?

(b) What is the probability that we will have at least 2 white flowers?

(c) What is the expected number of white flowers?

We can model this problem as a sequence of 20 Bernoulli trials, with "success" defined as having a white flower.

Since we are looking for the number of successes in a fixed number of trials, this is a binomial distribution. Let $X \sim$ Binom $(20, 1/4)$

(a) Using the binomial pmf:

$$P(X=5) = \binom{20}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^{15} \approx 0.2$$

(b) Let A = "at least two white flowers." It is much easier here to consider $A^c = 0$ or 1 white flower

$$
\begin{aligned}
P(A^c) &= P(X=0) + P(X=1) \\
&= \binom{20}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{20} + \binom{20}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{19} \\
&= \left(\frac{3}{4}\right)^{20} + 20\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{19} \approx 0.024
\end{aligned}
$$

Thus we have $P(A) = 1 - P(A^c) = 0.976$

(c) The expected value of a binomial random variable is np. For this case, we have $n = 20$ and $p = 0.25$, thus

$$E(X) = np = (20)(0.25) = 5$$