LECTURE: PROBABILITY AND COUNTING (I)

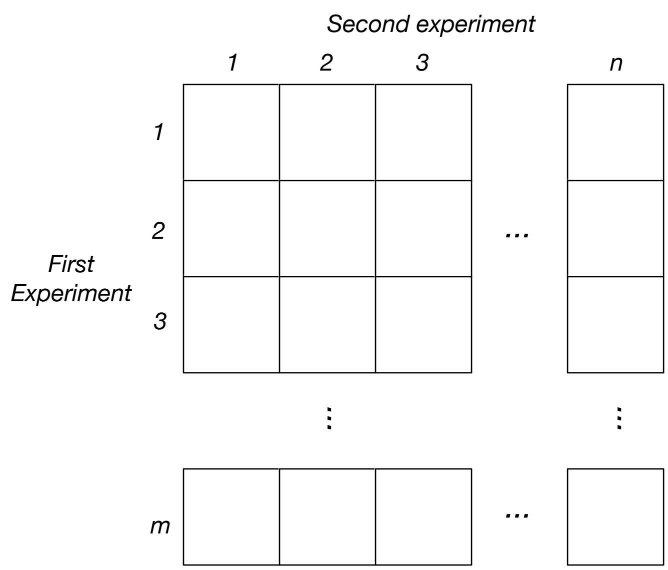
1. The Basic Principle of Counting

The mn-rule:

Suppose we are performing two experiments: Experiment 1 has m possible outcomes and Experiment 2 has n possible outcomes

Then there are mn possible outcomes for the two experiments.

To see this, draw a grid of boxes with the m outcomes from the first experiment on the left and the n outcomes of the second experiment across the top. There are mn total boxes, which are all the possible outcomes of both experiments.



Example 1:

If you roll 2 dice, then the total number of outcomes is $6 \cdot 6 = 36$.

If you roll 3 dice, then the total number of outcomes is

$$6 \cdot 6 \cdot 6 = 6^3 = 216$$

We can also think of this in terms of choosing items from groups: If there are m items in Group 1 and n items in Group 2, there are mn pairs of items consisting of one item from each group.

Example 2:

Suppose that license plates in the state of Peyamerica are composed of three letters (A-Z) followed by three digits (0-9).

(a) How many possible license plates are there?

We are choosing items from 6 groups. The first three groups contain 26 items, and the last three groups contain 10.

Thus the number of possibilities is: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$

(b) If the Peyamerica DMV (PMV) decided that letters and numbers could not be repeated, how many possible license plates would there be?

For the letters, the first group has 26 items. The second group only has 25 items, since we cannot choose the letter we chose from the first group. The third group has 24 items, since we cannot choose the letter we chose from the first two groups. The digits are similar.

Thus the number of possibilities is: $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11,232,000$

2. Permutations

Definition:

A **permutation** is an arrangement of distinct symbols where the order matters

All this means is that ABC $\neq$ CAB

Example 3:

How many different permutations are there of the letters P, I, and E?

Method 1: In this case, we can actually write out all the posibilities.

$$\text{PIE, PEI, EIP, EPI, IEP, IPE}$$

From this we see that there are 6 possible permutations.

Method 2: Use the method from before.

For the first letter, we have 3 letters to choose from; for the second, we have only 2; and for the third, there is only one remaining letter. Thus the number of permutations is:

$$3 \cdot 2 \cdot 1 = 3! = 6$$

We can generalize this:

Fact:

The number of permutations of n distinct symbols is $n!$

Example 4:

Suppose you're hosting a pie-eating contest. Gold, silver, and bronze medals are given to the top three finishers. If 100 contestants enter the contest, how many possibilities are there for the winners?

For the gold medal, we have 100 contestants to choose from. Since you cannot win more than one medal, we choose from 99 contestants for the silver medal and 98 contestants for the bronze medal. The number of medal possibilities is:

$$100 \cdot 99 \cdot 98 = 970,200$$

We can generalize this as well:

Fact:

The number of permutations of $r = 3$ items drawn from a group of $n = 100$ distinct items is:

$$nPr = n(n-1)(n-2)\cdots(n-(r-1)) = \frac{n!}{(n-r)!}$$

In terms of the previous example, we stop at $n-(r-1) = 100-2 = 98$ and not at $n-r = 97$

To get the second formula, notice that

$$n(n-1)(n-2)\cdots(n-r+1) = n(n-1)(n-2)\cdots(n-r+1)\frac{(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

3. Combinations

Definition:

A **combination** is an arrangement of distinct symbols where the order *doesn't matter*

Example 5:

100 people buy raffle tickets. Three numbers are chosen from a hat at random to win a free pie at the Peyameatery. How many possibilities are there for the winners?

This is *different* from the previous problem because the three prizes are identical, as opposed to the three distinct medals above.

So here the order doesn't matter and we have to use combinations. Let's look at this in a few stages:

STEP 1: Suppose for a second that the order mattered.

Then there would be $100 \cdot 99 \cdot 98 = 970,200$ possibilities of winners.

STEP 2: On the other hand, how many ways can we choose the winning numbers $1, 2, 3$ from the hat?

We can write out all the permutations:

$$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$$

This gives us a total of $3! = 6$ permutations.

STEP 3: Each of those 6 permutations corresponds to the same winners, so all we need to do is divide the number of permutations by 6.

This makes sense, because for our purposes $(1, 2, 3)$ and $(3, 2, 1)$ are the same, they give us the same winners.

Answer: The total number of possibilities for the winners is:

$$\frac{100 \cdot 99 \cdot 98}{6} = 161,700$$

Another way to think about this is grouping the $970,200$ possibilities in terms of packs of 6, and choosing a winner from every pack.

Fact:

The number of combinations of r items drawn from a group of n distinct items is:

$$\binom{n}{r} = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}$$

In other words, this is just the number nPr of permutations divided by the number $r!$ of the arranged objects.

Example 6:

The Brown crossword puzzle club consists of 5 undergraduate and 7 graduate students.

(a) How many different committees of 2 undergrads and 3 graduate students can be formed?

Since the order doesn't matter here, we need to use combinations.

There are $\binom{5}{2}$ possible groups of 2 undergrads, and $\binom{7}{3}$ possible groups of 3 graduate students. We multiply these together (by the mn-principle) to get:

$$\binom{5}{2}\binom{7}{3} = \frac{5!}{2!3!}\frac{7!}{3!4!} = \frac{5 \cdot 4}{2 \cdot 1}\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 10 \cdot 35 = 350$$

(b) What if two of the graduate students refuse to be on the same committee?

There are $\binom{7}{3} = 35$ possible groups of 3 graduate students.

Here it is easier to use the complementary event: How many contain both of the two students who refuse to serve together?

To make a three-person committee which includes these two, you need to include the two rival students plus one other student selected from the five remaining graduate students. Thus there are $\binom{2}{2}\binom{5}{1} = 5$ committees which include the two rival students.

Subtracting from 35, there are 30 committees which don't include both rival students.

As above, we multiply to get $10 \cdot 30 = 300$ possible committees.