LECTURE: PROBABILITY AND COUNTING (II)

1. Poker Face

Video: The Math of Poker

Example 1:

You are dealt a 5-card poker hand. What is the probability of getting a full house? (= 3 cards of one number plus 2 cards of another number)

Since the order doesn't matter, you have to use combinations.

Total Outcomes: There are 52 cards in a poker deck and a poker hand is 5 cards, there are $\binom{52}{5}$ possible poker hands.

Favorable Outcomes:

STEP 1: For the three-of-a-kind, choose 1 of the 13 numbers (say 7) and then 3 out of 4 cards with that number (7 of hearts, 7 of clubs...) This gives $\binom{13}{1}\binom{4}{3}$ choices for the three-of-a-kind.

STEP 2: For the pair, choose 1 of the 12 remaining numbers (since we have already chosen one of the numbers for the three-of-a-kind). Now select 2 out of the 4 cards of this number. This gives $\binom{12}{1}\binom{4}{2}$ ways to choose the pair once the three-of-a-kind has been chosen.

STEP 3: Multiply all of these together to get $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$ poker hands which are full houses.

$$\textbf{Answer: } \frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} \approx 0.00144$$

Wrong Method: Choose two numbers (say 5 and 7), then choose the three-of-a-kind and choose a pair, which gives $\binom{13}{2}\binom{4}{3}\binom{4}{2}$. This fails because this method does not distinguish between, say, `33366` and `33666`, and these are two distinct full houses. That said, if you multiply this by 2, you do get the correct answer.

2. String Problems

Example 2:

Consider the binary string `11000`. How many distinct orderings are there of the digits in this string? (like `01010` or `01100`)

This is like saying "You choose 2 undergrads from a group of 5" where 1 means you're chosen and 0 means you're not chosen

$$\text{Answer: } \binom{5}{2} = \frac{5!}{2!3!} = 10$$

Example 3:

A communication system has n antennas arranged in a line. Exactly m out of the n antennas are defective. What is the probability that the system will be functional (= no two consecutive antennas are defective)?

If `1` means functional and `0` means defective, then we want to avoid strings like `10011`

Total outcomes: Similar to above, we want to place m zeros in a string of length n, so there are $\binom{n}{m}$ possible arrangements.

Favorable Outcomes: Each functional system has the form

$$\_\_\, 1 \,\_\_\, 1 \,\_\_\, 1 \,\_\_\, 1 \,\_\_$$

Where in each space we either put one 0 or nothing. For example, `1101011` and `010110` are of this form but `11001` (not functional) is not

Since there are $n - m$ functional antennas, it really boils down to "In how many ways can we put m zeros in $(n - m) + 1$ slots?"

Thus there are $\binom{n-m+1}{m}$ favorable outcomes

$$\text{Therefore the probability is } \frac{\binom{n-m+1}{m}}{\binom{n}{m}}$$

Example: For $n = 4$ and $m = 2$, which we did before, we get

$$\frac{\binom{n-m+1}{m}}{\binom{n}{m}} = \frac{\binom{3}{2}}{\binom{4}{2}} = \frac{3}{6} = \frac{1}{2}$$

3. Box and Star Diagrams

Example 4:

Suppose you want to buy 10 donuts from Dunkin Donuts. There are five different flavors to choose from: Apple (A), Banana (B), Coconut (C), Dutch Caramel (D), and Elderflower (E). How many different assortments of donuts can you buy? Assume there are at least 10 of each kind of donuts in the store.

This problem uses a technique called **box and star diagrams**

Setting: Imagine you buy the donuts in a special box. They are arranged in a line, by flavor. There are four dividers, one between each donut type, and if a donut type is not present, we just put the dividers next to each other.

Here are some examples. Donuts are designated by $\star$ and dividers by $|$

$\star\star\,|\,\star\,|\,\star\star\star\,|\,\star\star|\,\star\star$	2 A, 1 B, 3 C, 2 D, 2 E
$\star\star\star||\,\star\star\star\,||\,\star\star$	3 A, 3 C, 2 E
$\star\star\star\star\star\star\star\star\star\star\,||||$	10 A

From the picture, we see that an assortment of 10 donuts can be represented as a linear string of 14 symbols: 10 $\star$ representing the donuts and 4 $|$ representing the dividers.

The number of assortments is the same as the number of ways of placing the four $|$ in 14 slots:

$$\text{Answer: } \binom{14}{4} = 1001$$

4. Binomial Theorem

Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

The coefficients $\binom{n}{r}$ are often called *binomial coefficients*.

Example 5:

$$\begin{aligned}(x+y)^4 =& \binom{4}{0}x^4y^0 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + \binom{4}{4}x^0y^4 \\ =& x^4 + 4x^3y + 2x^2y^2 + 4xy^3 + y^4\end{aligned}$$

Proof: We will illustrate this with the case $n = 3$

Suppose you're trying to expand out

$$\begin{aligned}(x+y)^3 =& (x+y)(x+y)(x+y) \\ =& xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ =& x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

What is the coefficient of xy^2? Notice it's the same thing as asking "In how many ways can we put two $y's$ in three slots, like $\underline{y}\,\underline{x}\,\underline{y}$. The answer is precisely $\binom{3}{2}$

Similarly, the coefficient of $x^{n-r}y^r$ is putting r y's in n slots, which is $\binom{n}{r}$

5. Multinomial Coefficients

The last counting technique is with multinomial coefficients. There is a general formula (see below), but it's easiest done directly

Example 6:

A group of 15 international relations students is divided into three groups. Each group consists of 5 students and is assigned a different country (Australia, Belgium, and China) on which to do a group final presentation. How many such groupings are possible?

STEP 1: Suppose the order matters, then there are 15! different arrangements of the students

STEP 2: There are 5! possible ways to rearrange the students in Australia, 5! ways to rearrange the students in Belgium, and 5! possible ways to rearrange the students in China

So the total number of groupings is

$$\frac{15!}{5!\,5!\,5!} = 756,756$$

Note: Alternatively, you could first choose 5 students out of 15 for the first group, then 5 students out of the remaining 10 for the second group, and then 5 students out of 5 for the last group, and get

$$\binom{15}{5}\binom{10}{5}\binom{5}{5} = 756,756$$

Definition:

The **multinomial coefficient is**

$$\binom{15}{5\;5\;5} = \frac{15!}{5!\,5!\,5!}$$

Fact:

The number of ways of dividing n objects into k distinct groups containing $n_1, \ldots, n_k$ objects each, where $n = n_1 + \cdots + n_k$ and each object appears in exactly one group, and is given by:

$$\binom{n}{n_1\ n_2\ \cdots\ n_k} = \frac{n!}{n_1!\, n_2!\ \cdots\ n_k!}$$

Example 7:

Now suppose we are dividing a group of 15 international relations students into three groups of 5 students each for a final project of the each group's choosing. How many different arrangements are possible?

In the previous example, the three groups were distinct, so it matters which one a student is assigned to.

In this case, the three groups are identical, so ABC = BCA etc. Since the order of the three groups does not matter, we divide the previous result by 3!

$$\frac{\binom{15}{5\ 5\ 5}}{3!} = \frac{15!}{5!\,5!\,5!\,3!} = 126,126$$

6. Summary: Counting Techniques

Case 1: The order matters

Case 1a: Repetition is allowed

Use n^k (think rolling 3 dice, which gives 6^3)

Case 1b: Repetition is not allowed

Use $nPr = n(n-1)\cdots(n-(r-1))$

(think license plates with different digits)

Case 2: The order doesn't matter

Case 2a: Repetition is allowed

Use box-and-star diagram (like the donut problem)

Case 2b: Repetition is not allowed

Use $\binom{n}{k}$ (think forming undergraduate groups)

For n objects in k groups then use $\binom{n}{n_1\, n_2 \cdots\, n_k}$