LECTURE: CONDITIONAL PROBABILITY

1. Warm-Up

Example 1:

How many different, distinct rearrangements are there of the word `MISSISSIPPI`?

STEP 1: Suppose the order matters, then there are 11! possible orderings of the 11 letters in `MISSISSIPPI`

STEP 2: There are 1! ways of ordering the M, 4! ways of ordering the I, 4! ways of ordering the S and 2! ways of ordering the P and therefore the total number of distinct rearrangements is:

$$\frac{11!}{1!\,2!\,4!\,4!} = 34,650$$

2. Conditional Probability

Sometimes the probability of an event will depend on whether or not another event has occurred. For example, the probability of rain tomorrow depends on whether or not it is raining today, especially during the tropical storm season. Or the probability of developing lung cancer depends on factors such as smoking history.

As usual, let S be our sample space and A and B be events

Definition:

$$P(A|B) = \text{ the (conditional) probability of } A \text{ given } B$$

Example 2:

Consider rolling a fair die. What is the probability of rolling a 1 given the die roll is odd?

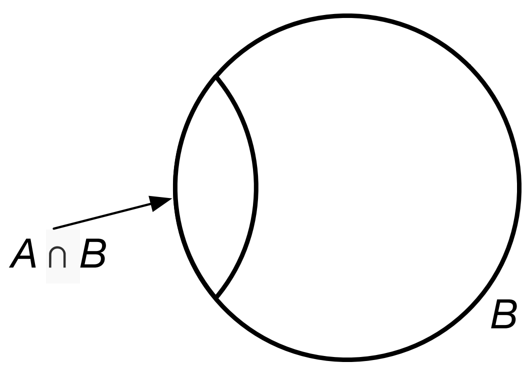
There are 3 possible odd rolls: 1, 3, and 5. Each of the three odd rolls is equally likely. Thus the probability of rolling a 1 given an odd roll is $\frac{1}{3}$

Note that what we did above is basically reduce our sample space from $S =$ all six possible rolls to $B =$the three odd rolls, as if B were our new universe.

In fact, another definition of conditional probability, which is more useful in theory purposes, is the following:

Fact:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



This makes intuitively sense because if our sample space is B, then $A \cap B$ is the part of A that lies in B. Furthermore, we divide by $P(B)$ to ensure that $P(B|B) = 1$ (the probability of B given B)

Example 3:

You roll two standard, fair six-sided dice. What is the the probability that the sum is greater than or equal to 11, given that the first roll is a 6?

Let A = "the sum is greater than or equal to 11" and B = "the first roll is a 6"

Method 1: Given the the first roll is a 6, only 2 of the 6 possibilities have a sum of dice greater than or equal to 11, namely $(6, 5)$ and $(6, 6)$. Thus the conditional probability is $P(A|B) = 2/6 = 1/3$.

Method 2: $P(B) = 6/36$. Here $A \cap B$ (first roll is 6 and sum is ≥ 11) has two elements: $(6, 5)$ and $(6, 6)$ so $P(A \cap B) = 2/36$. Therefore

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{6/36} = \frac{1}{3}$$

3. More Practice

Example 4:

In the card game bridge, 52 cards are dealt out equally to 4 players (called North, South, East and West). Given that North and South have a total of 8 spades between them (out of 13 total spades), what is the probability that East has 3 spades?

If North and South have 26 cards and 8 spades between them, then East and West much have the remaining 26 cards and 5 spades between them.

The reduced sample space we will work with is the sample space of possible hands for East.

Total Outcomes: Since East has 13 of the remaining 26 cards, there are $\binom{26}{13}$ possible hands for East.

Favorable outcomes: In how many of those hands does East have exactly 3 spades?

STEP 1: First choose 3 of the remaining 5 spades. There are $\binom{5}{3}$ ways to do this.

STEP 2: Then we need to choose the non-spade cards. East has 10 non-spade cards, and there are $26 - 5 = 21$ non-spade cards to choose from, thus there are $\binom{21}{10}$ ways to choose these.

STEP 3: Multiplying these together, there are $\binom{5}{3}\binom{21}{10}$ possible hands where East has exactly 3 of the remaining spades.

$$\text{Answer: } \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} \approx 0.339$$

Example 5:

Café Peyam sells 4 Apple, Blueberry, Chocolate, and Yam Pies. How many different combinations of 20 pies contain at least 8 Apple Pies and at most 5 Blueberry Pies?

$$\text{A} \quad \text{A} \quad \text{A} \quad \text{A} \quad \text{A} \quad \text{A} \quad \text{A} \quad \text{A} \quad \underbrace{\star \star \star \cdots \star}_{12}$$

Given that the first 8 have to be Apple Pies, we just need to figure out:

"How many combinations of 12 pies contain at most 5 blueberry pies?"

Here it is easier to deal with the complementary event:

"How many combinations of 12 pies contain at least 6 blueberry pies?"

$$\text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \text{B} \quad \underbrace{\star \star \star \star \star \star}_{6}$$

Or, in other words "How many combinations of 6 pies are there?"

$$\star \star \mid \star \mid \star \star \mid \star$$

By the stars-and-bars method, there are $\binom{9}{3}$ choices of 6 pies

The total number of combinations of 12 pies is $\binom{15}{3}$

$$\text{Answer: } \binom{15}{3} - \binom{9}{3} = 455 - 84 = 371$$

4. Independence

Two events A and B are independent if knowledge of one event does not affect the other one.

Here are some intuitive examples:

(1) You are flipping coins repeatedly. Each coin flip is independent of all other coin flips, since there is no way for coin flips to

affect each other. Now imagine you flip 10 heads in a row. It is tempting to say that the streak of heads must break, and that it is more likely than not that the 11th flip is a tail. This is known as the **gambler's fallacy** and is false since the flips are independent, and each flip has a 1/2 probability of heads.

(2) In blackjack, if a player is dealt an ace, the next deal is less likely to be an ace, since the cards are drawn without replacement. Successive deals of cards are thus not independent. This is the basis for counting cards in blackjack and other games.

(3) "Smoking" and "Contracting lung cancer" are not independent since a causal link has been shown between the two.

Definition:

Two events A and B are *independent* if

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

In other words, knowledge of one does not affect the probability of the other.

The second definition below is much easier to deal with in practice:

Definition:

A and B are *independent* if and only if

$$P(A \cap B) = P(A)P(B)$$

Why? Simply use

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$$

So if, for instance, $P(A|B) = P(A)$ then we get $P(A \cap B) = P(A)P(B)$