LECTURE: INDEPENDENCE

1. Independence

Definition:

A and B are **independent** if

$$P(A \cap B) = P(A)P(B)$$

Example 1:

Roll a single die and consider the following events

A: we roll an odd number

B: we roll an even number

C: we roll a 1 or a 2

(a) Are A and B independent?

(b) Are A and C independent?

(a) Because $A \cap B = \emptyset$, $P(A \cap B) = 0$ so

$$P(A \cap B) = 0 \neq P(A)P(B) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

It makes sense that they're not independent, because if A occurs then B cannot occur, so knowledge of A influences the outcome of B

Moral: Do not confuse independence with disjointness

(b) Notice that $P(A|C) = 1/2$. Since $P(A) = 1/2$ as well, $P(A|C) = P(A)$ thus A and C are independent.

Example 2:

This time roll two dice and consider the following events:

A: we roll a 4 on the first die

B: the sum of the two dice is 6

C: the sum of the two dice is 7

(a) Are A and B independent?

(b) Are A and C independent?

(c) Are B and C independent?

(a) $P(A \cap B) = 1/36$, since $A \cap B$ consists of just $(4, 2)$.

Moreover $P(A) = 1/6$ whereas B consists of $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$ thus $P(B) = 5/36$.

$$P(A)P(B) = (1/6)(5/36) = 5/216 \neq 1/36 = P(A \cap B)$$

So A and B are not independent.

(b) We have again that $P(A \cap C) = 1/36$, since if we know the first die is 4 and the two dice sum to 7, the second die must be a 3; $P(A) = 1/6$. Finally $P(C) = 6/36 = 1/6$, since C comprises 6 simple events

$(1,6),(2,5)\cdots(6,1)$.

$$P(A)P(C) = (1/6)(1/6) = 1/36 = P(A \cap C)$$

And so A and C are independent.

Why does this make sense? No matter what we roll on the first die, there are still 6 possibilities of getting a sum of 7 when the second die is rolled.

(c) B and C are not independent because they are disjoint, just like above.

2. Multiplicative and Additive Laws

The multiplicative and additive laws of probability give the probabilities of intersections and unions of events, and are important in constructing probabilities for more complicated events.

Multiplicative Law:

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

If A and B are independent, then:

$$P(A \cap B) = P(A)P(B)$$

The additive law of probability gives the probability of the union of two events.

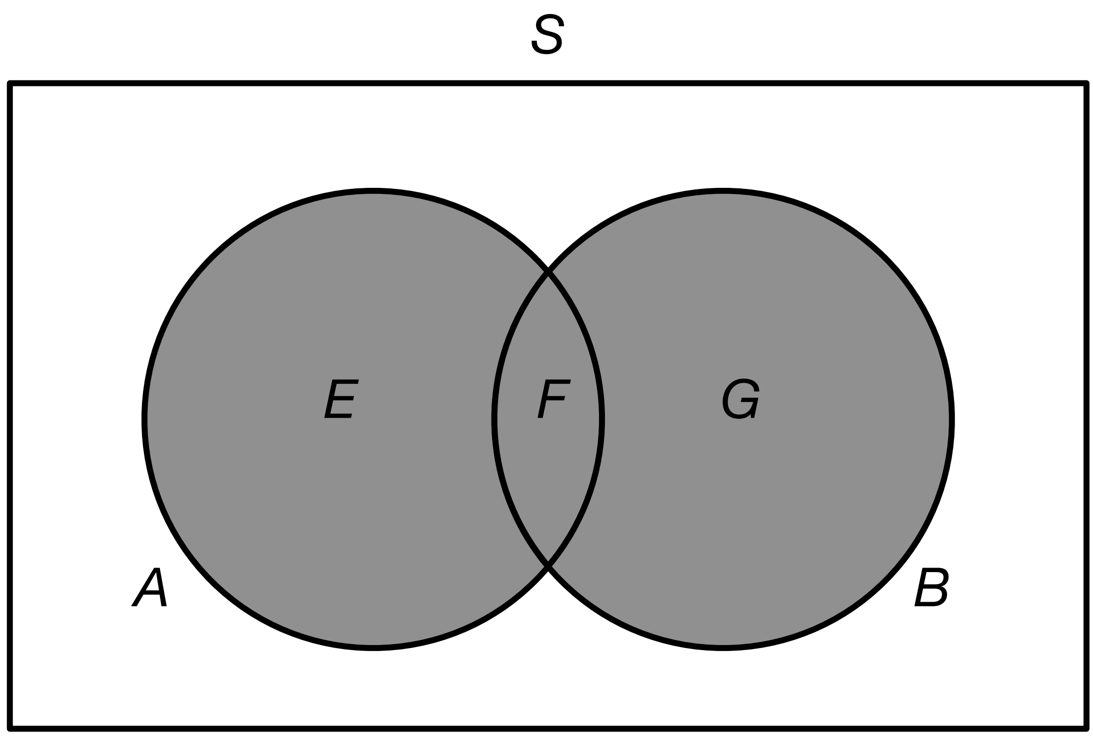
Additive Law:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are disjoint then:

$$P(A \cup B) = P(A) + P(B)$$

To see this is true, consider the Venn diagram below.



If we add the probability of A and B, we are "double-counting" $A \cap B$, so we have to subtract it off to get $P(A \cup B)$.

This can be extended to unions of more than two sets, although the procedure is more complicated.

3. Urn Problem

We can use the laws above in the following problem.

Example 3:

You have two urns of balls. The first urn contains 1 red ball and 3 white balls. The second urn contains 2 red balls and 2 white balls. You choose an urn at random and then draw a ball from the chosen urn.

(a) What is the probability that the ball you draw is red?

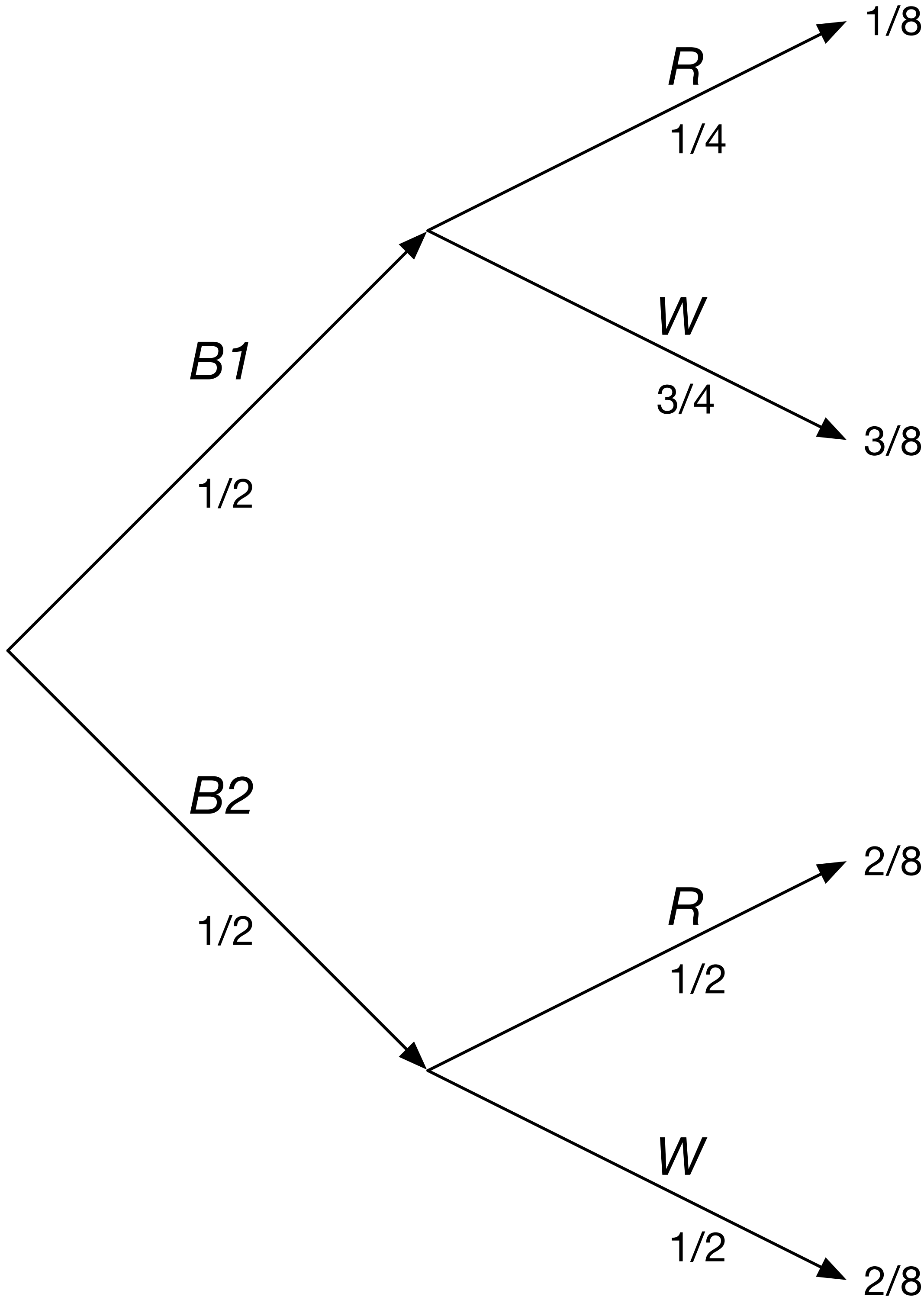
(b) What is the probability we chose urn 1 in the first step given that we draw a red ball?

(a) Here it is easier to draw a **tree diagram**.

Let B_1 be the event that we choose urn 1, and B_2 the event that we choose urn 2.

Let R be the event that we draw a red ball, and W be the event that we draw a white ball.

Then we have the following tree diagram.


1/8
R
1/4
W
3/4
3/8
B1
1/2
B2
1/2
2/8
R
1/2
W
1/2
2/8

The first two arrows from the left represent the probabilities of drawing from urn 1 and urn 2, which are $\frac{1}{2}$ respectively.

The first arrow with R (from the top) represents the probability of drawing a red ball given that you chose urn 1, which is $\frac{1}{4}$ because there is 1 red ball in urn 1 and 3 white balls

Finally, the $\frac{1}{8}$ represents $P(B_1 \cap R)$ which is the probability of choosing the first urn and drawing a red ball, which is

$$P(B_1 \cap R) = P(B_1)P(R|B_1) = (1/2)(1/4) = 1/8$$

Since there are two (disjoint) ways of drawing a red ball, either from urn 1 with probability $\frac{1}{8}$ or from urn 2 with probability $\frac{1}{4}$, we get

$$P(R) = 1/8 + 2/8 = 3/8$$

(b) Here are looking for the conditional probability $P(B_1|R)$. We computed $P(R)$ in the first part, and we can read off $P(B_1 \cap R) = 1/8$ from the tree. Thus by the definition of conditional expectation:

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)} = \frac{1/8}{3/8} = \frac{1}{3}$$

4. Bayes' Rule

Given two events A and B, Bayes' rule is a mathematical formula for calculating $P(A|B)$ given $P(B|A)$

Bayes' Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$\textbf{Why?}\ P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \cap A)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

Example 4:

In the context of the previous problem, is the probability that you choose Urn 1 in the first step given that the ball drawn is red?

We are looking for $P(B_1|R)$.

Here is is easier to figure out $P(R|B_1) = 1/4$ since we know exactly what happens if we draw from Urn 1, which

Hence by Bayes' Rule:

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R)} = \frac{(1/4)(1/2)}{3/8} = \frac{1}{3}$$