HOMEWORK 10 – SOLUTIONS

Problem 1:

Define

$$g_n(x) = \begin{cases} 2n + 4n^4 x & \text{if } x \in \left[-\frac{1}{2n^3}, 0\right] \\ 2n - 4n^4 x & \text{if } x \in \left(0, \frac{1}{2n^3}\right] \\ 0 & \text{otherwise.} \end{cases}$$

Note the graph of g_n looks like a spike on $[-\frac{1}{2n^3}, \frac{1}{2n^3}]$ which reaches its maximum height $2n$ at 0. As n increases, the spike gets thinner and taller.

Now let

$$f(x) = \begin{cases} g_n\left(-\frac{1}{2n^3} + x - n\right) & \text{if } x \in [n, \frac{1}{n^3}], n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, we are putting smaller and thinner spikes at each n for all natural numbers n. Note f is continuous because each g_n is continuous and $g_n(\pm\frac{1}{2n^3}) = 0$, and f is nonnegative because each g_n is nonnegative.

The idea is that the spikes in the graph of f grow taller, so that $\limsup_{x\to\infty} f(x) = \infty$, but their integrals grow smaller fast enough that $\int_{\mathbb{R}} f < \infty$. To show this carefully, note $f(n + \frac{1}{2n^2}) = g_n(0) = 2n \to \infty$ as $n \to \infty$, implying $\limsup_{x\to\infty} f(x) = \infty$. Finally,

$$\int_{\mathbb{R}} f = \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n = \sum_{n=1}^{\infty} 2n \cdot \frac{1}{n^3} \cdot \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

We conclude f is the desired function.

Problem 2:

Part (a): WLOG assume $f \geq 0$. Here we prove the contrapositive, i.e. we will show that if f is not 0 almost everywhere, the integral of f must be positive. Let $E = \{x \in X : f(x) > 0\}$ be the set on which f is nonzero. We then employ the following useful decomposition

$$E = \bigcup_{n=1}^{\infty} E_n \qquad\qquad E_n = \left\{ x \in X : f(x) > \frac{1}{n} \right\}.$$

If f is not 0 almost everywhere, then one of the sets E_n has positive measure, i.e. $\mu(E_n) = r > 0$ for some n. But then we have

$$f \geq \frac{1}{n}\chi_{E_n},$$

from which it follows that

$$\int f \geq \int \frac{1}{n}\chi_{E_n} = \frac{1}{n}\mu(E)n = \frac{r}{n} > 0.$$

Part (b): Again we prove the contrapositive: we will show that if there is a set of positive measure on which $f(x) \neq 0$, then there exists measurable E such that $\int_E f(x)dx \neq 0$.

Write

$$\{x : f(x) \neq 0\} = \{x : f(x) > 0\} \cup \{x : f(x) < 0\}.$$

Since $\{x : f(x) \neq 0\}$ has positive masure, one of these sets on the right must as well; WLOG assume $\{x : f(x) > 0\}$ has positive measure. Now write

$$\{x : f(x) > 0\} = \bigcup_{k=1}^{\infty} \left\{ x : f(x) > \frac{1}{k} \right\}$$

At least one of the sets on the right must have positive measure, so there is k and some positive measure set E on which $f(x) > 1/k$. Then

$$\int_E f \geq \int_E \frac{1}{k} = \frac{1}{k}\mu(E) > 0.$$

Problem 3:

STEP 1: We will first show that $\int_0^1 \frac{1}{r^p} dr$ converges if and only if $p < 1$, and also $\int_1^\infty \frac{1}{r^p} dr$ converges if and only if $p > 1$.

For $p > 1$:

$$\int_0^1 \frac{dx}{x^p} = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{x^p} = -\frac{1}{p-1} + \lim_{\epsilon \to 0} \frac{1}{(p-1)\epsilon^{p-1}} = +\infty$$

For $p = 1$:

$$\int_0^1 \frac{dx}{x} = \lim_{\epsilon to 0} \int_\epsilon^1 \frac{dx}{x} = \lim_{\epsilon \to 0}(-\log \epsilon) = +\infty.$$

For $p < 1$:

$$\int_0^1 \frac{dx}{x^p} = \lim_{\epsilon \to 0} \int_\epsilon^1 \frac{dx}{x^p} = -\frac{1}{p-1} + \lim_{\epsilon \to 0} \frac{1}{(p-1)\epsilon^{p-1}} = 0.$$

The results for the bounds 1 to ∞ are obtained by noting

$$\int_1^\infty \frac{dx}{x^p} = \int_0^1 \frac{dx}{x^{2-p}}.$$

STEP 2: Using the polar coordinate transform

$$\begin{aligned}\int_{\mathbb{R}^d} |f_a(x)|dx &= \int_0^\infty \left(\int_{|x|=r} f(x)dS(x)\right) dr \\ &= \int_0^1 \left(\int_{|x|=r} \frac{1}{r^a} dS(x)\right) dr \\ &= \int_0^1 \left(\frac{1}{r^a} C(d) r^{d-1}\right) dr \\ &= C(d) \int_0^1 \frac{1}{r^{a-d+1}} dr.\end{aligned}$$

By the first step, f_a is therefore integrable if and only if $a < d$.

By a similar computation,

$$\int_{\mathbb{R}^d} |g_a(x)|dx = C(d) \int_1^\infty \frac{1}{r^{a-d+1}} dr,$$

so by the first step g_a is integrable if and only if $a > d$.

Problem 4:

We want to show that if we have nonnegative measurable functions a_k, then

$$\int \sum_{k=1}^\infty a_k(x)dx = \sum_{k=1}^\infty \int a_k(x)dx.$$

Let $f_n(x) = \sum_{k=1}^n a_k(x)$, so the f_n form a nondecreasing sequence nonnegative functions. Then we can apply the monotone convergence theorem:

$$\int \sum_{k=1}^\infty a_k(x)dx = \int \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int f_n(x)dx = \sum_{k=1}^\infty \int a_k(x)dx.$$

Problem 5:

STEP 1: Suppose $\{f_n\}$ is Cauchy in L^1, that is $\|f_n - f_m\|$ goes to 0 as $m, n \to \infty$. The plan is to extract a subsequence of $\{f_n\}$ that converges to some f pointwise *and* in the norm. This can be achieved if the convergence is fast enough.

STEP 2: Claim # 1: There is subsequence $\{f_{n_k}\}$ such that

$$\left\|f_{n_{k+1}} - f_{n_k}\right\| \leq 2^{-k}$$

Proof of Claim: You do this inductively. Suppose you found f_{n_k}, then by Cauchiness with $\epsilon = 2^{-k}$ there is $N = N(2^{-k})$ such that if $n \geq N$ then

$$\|f_n - f_{n_k}\| \leq 2^{-k}$$

Then just let $n_{k+1} = N$ ✓

STEP 3: Our function f

$$\text{Define: } f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$

$$\text{And } g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$$

$$\begin{aligned}\int g\,dx &= \int |f_{n_1}| + \int \sum_{k=1}^{\infty} \left|f_{n_{k+1}} - f_{n_k}\right| dx \\ &= \int |f_{n_1}| + \sum_{k=1}^{\infty} \int \left|f_{n_{k+1}} - f_{n_k}\right| dx \\ &\leq \int |f_{n_1}| + \sum_{k=1}^{\infty} 2^{-k} \\ &= \int |f_{n_1}| + 1 < \infty\end{aligned}$$

(The interchange of series and integrals is justified by the problem about series)

Hence g is integrable, and since $|f| \leq g$, this implies f is integrable.

In particular, the series defining f converges almost everywhere, and since the partial sums of that series are precisely f_{n_k} (telescoping series), we find that $f_{n_k} \to f$ a.e. x

STEP 4: Claim # 2: $f_{n_k} \to f$ in L^1

This follows because each partial sum is dominated by g. Therefore, by the Dominated Convergence Theorem, we get $\|f_{n_k} - f\| \to 0$

STEP 5: Claim # 3: $f_n \to f$ in L^1

Just need to use Cauchiness: Given $\epsilon > 0$ there is N such that for all $m, n > N$ then $\|f_n - f_m\| < \frac{\epsilon}{2}$. If n_k (for k large enough) is chosen such that $n_k > N$ and $\|f_{n_k} - f\| < \frac{\epsilon}{2}$ (from **STEP 4**) then if $n > N$ we have

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence f_n converges to f in L^1 $\square$

Problem 6:

Let $\epsilon > 0$ be given, then since $m(E) < \infty$, by Egorov, there is $A_\epsilon \subseteq E$ with $f_n \to f$ uniformly on A_ϵ and $m(E - A_\epsilon) < \epsilon$.

By uniform conv, there is N such that if $n > N$ then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A_\epsilon$, but then

$$\begin{aligned}
\int_E |f - f_n|\, dx &= \left(\int_{A_\epsilon} + \int_{E-A_\epsilon}\right) |f_n - f| \\
&= \int_{A_\epsilon} \underbrace{|f_n(x) - f(x)|}_{<\epsilon}\, dx + \int_{E-A_\epsilon} \underbrace{|f_n(x) - f(x)|}_{\leq 2M}\, dx \\
&\leq m(A_\epsilon)\, \epsilon + 2M\, m(E - A_\epsilon) \\
&\leq m(E)\epsilon + 2M\epsilon = \epsilon\,(m(E) + 2M) \quad \square
\end{aligned}$$

Problem 7:

Part (a):

$$\begin{aligned}
\int f &= \int_{\{x|\, f(x)>t\}} f + \int_{\{x|\, f(x)\leq t\}} f \\
&\geq \int_{\{x|\, f(x)>t\}} f \\
&\geq \int_{\{x|\, f(x)>t\}} t \\
&= tm\,\{x \mid f(x) > t\}
\end{aligned}$$

$$\text{Hence } tm\,\{x \mid f(x) > t\} \leq \int f$$

Part (b):

$$\begin{aligned}\int |f|^p &= \int_{\{x|\ |f(x)|>t\}} |f|^p + \int_{\{x|\ |f(x)|\leq t\}} |f|^p \\ &\geq \int_{\{x|\ |f(x)|>t\}} |f|^p \\ &\geq \int_{\{x|\ |f(x)|>t\}} t^p \\ &= t^p m\,\{x \,|\, |f(x)| > t\}\end{aligned}$$

$$t^p m\,\{x|\ \ |f(x)| > t\} \leq \int |f|^p$$

And dividing by $t^p > 0$ gives us the result

Problem 8:

$$\int_X |T(f)|^p \, d\mu = \int_X |f|^p \, |g|^p \, d\mu \leq \int_X |f|^p \, \|g\|^p_{L^\infty} \, d\mu = \|g\|^p_{L^\infty} \int_X |f|^p \, d\mu$$

In particular, taking $p-$th root of both sides we get

$$\|T(f)\|_{L^p(X)} \leq \|g\|_{L^\infty(X)} \, \|f\|_{L^p(X)}$$

In particular $T : L^p \to L^p$ is bounded and $\|T\| \leq \|g\|_{L^\infty(X)}$

Problem 9:

WLOG assume $f \geq 0$ and let $\epsilon > 0$ be given.

Let $f_n =: f\,\chi_{E_n}$ where $E_n = \{x \,|\, f(x) \leq n\}$ and notice $f_n \leq n$

Then $f_n \geq 0$ measurable and $f_n \nearrow f$, so by the Monotone Convergence Theorem we have $\lim_{n\to\infty} \int f - f_n = 0$, so there is $N > 0$ such that

$$\int f - f_N < \frac{\epsilon}{2}$$

Let δ TBA, then if $m(E) < \delta$, then

$$\begin{aligned}\int_E f &= \int_E \underbrace{f - f_N}_{\geq 0} + \int_E f_N \\ &\leq \int_{\mathbb{R}^d} f - f_N + \int_E \underbrace{f_N}_{\leq N} \\ &\leq \frac{\epsilon}{2} + N m(E) \\ &< \frac{\epsilon}{2} + N\delta\end{aligned}$$

If you choose δ such that $N\delta < \frac{\epsilon}{2}$ then you get $\int_E f < \epsilon$ □

Problem 10:

Define $f = 0$ and

$$f_1 = \chi_{[0,1]}$$
$$f_2 = \chi_{\left[0,\frac{1}{2}\right]}\ f_3 = \chi_{\left[\frac{1}{2},1\right]}$$
$$f_4 = \chi_{\left[0,\frac{1}{4}\right]}, f_5 = \chi_{\left[\frac{1}{4},\frac{1}{2}\right]}, f_6 = \chi_{\left[\frac{1}{2},\frac{3}{4}\right]}, f_7 = \chi_{\left[\frac{3}{4},1\right]}$$

Then $f_n(x) \nrightarrow 0$ for no x (Given x we have $f_n(x) = 1$ for infinitely many n), but

$$\|f_n - f\| = \int_0^1 |f_n(x) - f(x)|\, dx = \int_0^1 |f_n| \to 0$$

Since the areas under f_n shrink to 0.