ANALYSIS – HOMEWORK 10

Problem 1: Show there is a continuous function $f \geq 0$ on $\mathbb{R}$ with

$$\int_{\mathbb{R}} f(x)dx < \infty \text{ but } \limsup_{x \to \infty} f(x) = \infty$$

Problem 2:

(a) Show that if $\int_X |f| \, dx = 0$ then $f(x) = 0$ a.e.

(b) Show that if $\int_E f(x)dx = 0$ for every measurable E then $f(x) = 0$ a.e.

Problem 3: Consider

$$f_a(x) = \begin{cases} \frac{1}{|x|^a} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \text{and} \quad g_a(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ \frac{1}{|x|^a} & \text{if } |x| > 1 \end{cases}$$

Show that f_a is integrable precisely when $a < d$ and g_a is integrable precisely when $a > d$

Hint: Here is is useful to use the polar coordinates formula

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \left(\int_{|x|=r} f(x)dS(x) \right) dr$$

Here $S(x)$ is the surface measure on the sphere. You're also allowed to assume that the surface measure of $\{|x| = r\}$ is $C\, r^{d-1}$ where $C = C(d)$

Problem 4: Show that if $a_k(x) \geq 0$ are measurable functions, then

$$\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx$$

Problem 5: Show $L^1(\mathbb{R})$ is complete

Hint: First find a subsequence f_{n_k} with $\left\| f_{n_{k+1}} - f_{n_k} \right\| \leq 2^{-k}$ and let

$$f(x) =: f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x)$$

Problem 6: Prove the Bounded Convergence Theorem, assuming (f_n) are all supported on a set E with $\mu(E) < \infty$

Problem 7:

(a) Suppose $f \geq 0$ and f is integrable. Show that if $t > 0$ then

$$m\left\{x \mid f(x) > t\right\} \leq \frac{1}{t}\left(\int f\right)$$

(b) More generally, show if $f \in L^p\left(\mathbb{R}^d\right)$ with $1 \leq p < \infty$ then for every $t > 0$ we have

$$m\left\{x \mid |f(x)| > t\right\} \leq \frac{1}{t^p}\left(\int |f|^p \, dx\right)$$

Problem 8: If $g \in L^\infty(X)$ consider $T(f) = fg$. Show that this operator is bounded on L^p for $1 \leq p \leq \infty$ and that its operator norm is at most $\|g\|_{L^\infty}$

Problem 9: Show that if f is integrable, then for all $\epsilon > 0$ there is $\delta > 0$ such that if $\mu(E) < \delta$ then $\int_E |f(x)|\, dx < \epsilon$

Hint: Let $f_n =: f\chi_{E_n}$ where $E_n = \{x \mid f(x) \leq n\}$

Problem 10: Show that there is a sequence $\{f_n\}$ with $f_n \in L^1$ and a function f such that $f_n \to f$ in L^1 but $f_n(x) \to f(x)$ for no x