LECTURE: LEBESGUE INTEGRAL

Setting: Suppose you're given a measure space $(X, \mathcal{M}, \mu)$

So X is a set, $\mathcal{M}$ is the family of measurable subsets of X and μ is a measure on $\mathcal{M}$

We will now define the Lebesgue integral of real-valued (measurable) functions f on X. This is done in four steps:

1. Step 1: Simple Functions

Definition:

If E is any set then the **characteristic function** of E is

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Notice $\chi_E(x)$ is a measurable function if and only if E is measurable

Definition:

A **simple function** is a function of the form

$$\phi(x) = \sum_{k=1}^{n} y_k \, \chi_{E_k}(x)$$

Where $E_k \in \mathcal{M}$ and y_k are real numbers.

WLOG, we can assume that the E_k are disjoint and each y_k is nonzero and distinct (otherwise just group sets with common value together)

In that case, the range of ϕ is the finite set $\{y_1, \ldots, y_n\}$, and also $E_k = f^{-1}(\{y_k\})$

Definition:

The **Lebesgue integral** of $\phi = \sum_{k=1}^{n} y_k \chi_{E_k}(x)$ is

$$\int_X \phi d\mu = \sum_{k=1}^{n} y_k \mu(E_k)$$

It can be shown that this definition is independent of the representation used, that is if

$$\phi(x) = \sum_{k=1}^{n} y_k \chi_{E_k}(x) = \sum_{k=1}^{m} z_k \chi_{F_k}(x) \text{ then } \sum_{k=1}^{n} y_k \mu(E_k) = \sum_{k=1}^{m} z_k \mu(F_k)$$

We can also define $\int_A \phi(x) dx$ where A is any measurable subset of X:

Definition:

$$\int_A \phi d\mu = \int_X \phi \, \chi_A d\mu = \sum_{k=1}^{n} y_k \mu(E_k \cap A)$$

This integral has the following nice properties:

Facts:

Let ϕ and ψ be simple functions, then

(1) (Linearity) $\int_X (\phi + c\psi) = \int_X \phi + c \int_X \psi$.

(2) (Monotonicity) If $\phi \leq \psi$, then $\int_X \phi \leq \int_X \psi$

(3) (Additivity) If E and F are disjoint then

$$\int_{E \cup F} \phi = \int_E \phi + \int_F \phi$$

(4) (Triangle Inequality) $|\phi|$ is simple and

$$\left| \int_X \phi \right| \leq \int_X |\phi|$$

(5) Define the real-valued function ρ on $\mathcal{M}$ by

$$\rho(E) = \int_E \phi d\mu$$

Then ρ is a (signed) measure on $\mathcal{M}$

The last one may seem a little strange, but is really useful for proving the Radon-Nikodym Theorem and gives a new measure on $\mathcal{M}$

Note: Compare this definition with the Riemann integral! In the Riemann integral we focused on the $x-$values of f, which we partitioned into pieces, but here we're focusing more on the $y-$values $y_1, y_2, \cdots, y_n$, we're partitioning the range of f

2. Step 2: Bounded Non-Negative Functions with Finite Support

Definition:

The **support** of f is $\text{supp}(f) = \{x \mid f(x) \neq 0\}$

(Sometimes it's defined as the closure of the above, but the distinction is not important here)

Definition:

f is **supported on** E if $f(x) = 0$ whenever $x \notin E$

In this second step, we're interested in bounded non-negative functions such that $\mu(\text{supp}(f)) < \infty$

(Visualize this like functions with compact support, functions that are 0 after a while)

The key to defining $\int_X f(x)$ here lies in the following

Simple Approximation Lemma:

Let $f : (X, \mathcal{M}) \to [0, \infty)$ be bounded, and of finite support.

Then there is an increasing sequence of nonnegative simple functions $\{\phi_n(x)\}$ i.e. $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$ such that $\phi_n \to f$ pointwise.

Proof-Idea: The idea is first of all to truncate f if it becomes too large, and then partition the range of those truncated functions into fine layers, like the lower sum in Riemann integrals.

We then define the integral in the following way:

Definition:

If f is non-negative, bounded, and of finite support then

$$\int_X f(x)dx =: \lim_{n\to\infty} \int_X \phi_n(x)dx$$

Where $\{\phi_n\}$ is any sequence of bounded simple functions with same support as f such that $\phi_n \to f$ pointwise

Problem: We don't know if this limit is independent of the sequence (ϕ_n) used, yet alone that it even exists, because $\phi_n \to f$ pointwise does **not** imply that $\int \phi_n \to \int f$. What saves us is:

Egorov's Theorem:

If $\mu(E) < \infty$ and $f_n : E \to \mathbb{R}$ is a sequence of functions with $f_n \to f$ pointwise on E. Then if $\epsilon > 0$ there is a closed subset $A_\epsilon \subseteq E$ with $\mu(E - A_\epsilon) < \epsilon$ such that $f_n \to f$ uniformly on A_ϵ

Using Egorov's Theorem you can show that in fact this limit exists and is independent of the sequence ϕ_n used

At this point, you can show that if f is Riemann Integrable on $[a, b]$ then the Riemann and Lebesgue integrals of f are the same

3. Step 3: Non-Negative Functions

Definition:

If f is ≥ 0 is measurable, then

$$\int_X f(x)dx =: \sup_g \int_X g(x)dx$$

Where the sup is taken over all g from **STEP 2** with $0 \leq g \leq f$

Definition:

$$f \text{ is } \textbf{integrable} \text{ if } \int_X |f(x)|\, dx < \infty$$

(We can actually remove the absolute value since $f \geq 0$)

The same properties (linearity, additivity, monotonicity, triangle inte-quality) hold.

Fact:

If f is integrable and $0 \leq g \leq f$ then g is integrable

Proof: Follows because $\int |g| \leq \int |f| < \infty$

Definition:

A property holds **almost everywhere** if it holds everywhere except for a set of measure 0

Fact:

If f is integrable then $f(x) < \infty$ for almost every x

Proof: Let $E_k = \{x \mid f(x) \geq k\}$ and $E_\infty = \{x \mid f(x) = \infty\}$ then

$$\int_X f(x)dx \geq \int_{E_k} f(x)dx \geq \int_{E_k} k = k\mu(E_k)$$

Hence $\mu(E_k) \leq \frac{1}{k}\int_X f(x)dx \to \infty$ and hence $\mu(E_k) \to 0$

But since $E_k \searrow E_\infty$ and $\mu(E_k) < \infty$ we get $\mu(E_\infty) = 0$

4. Step 4: General Case

For general f, just write $f = f^+ - f^-$ where

$$f^+ = \max\{f, 0\} \qquad f^- = \max\{-f, 0\}.$$

Here $f^\pm$ are non-negative functions and so

Definition:

$$\int_X f(x)dx =: \int_X f^+ - \int_X f^-$$

Definition:

f is **integrable** if $\int_X |f(x)|\, dx < \infty$

Note: $\int f$ is independent of the decomposition used:

If $f = f_1 - f_2 = g_1 - g_2$ where f_i and g_i are non-negative and measurable

Then $f_1 + g_2 = g_1 + f_2$ so by **STEP 3** we have

$$\int f_1 + \int g_2 = \int g_1 + \int f_2 \Rightarrow \int f_1 - \int f_2 = \int g_1 - \int g_2$$

All the facts discussed before (linearity, additivity, monotonicity, triangle inequality) are true here as well

Fact:

Suppose $\int_X |f|\,dx = 0$ then $f = 0$ almost everywhere

Finally, integrable functions enjoy the following property, called **absolute continuity**:

Fact:

If f is integrable then for all $\epsilon > 0$ there is $\delta > 0$ such that

$$\mu(E) < \delta \Rightarrow \int_E |f(x)|\,dx < \epsilon$$

5. L^p SPACES

The space of integrable functions has a particularly nice structure.

Definition:

$L^1(X)$ = space of integrable functions

Definition:

If f is integrable, then the L^1 **norm** of f is

$$\|f\| = \|f\|_{L_1} =: \int_X |f(x)|\,dx$$

You can check that this is a norm in the usual sense. For example we have $\|f+g\| \leq \|f\| + \|g\|$

Norms allow us to define the distance between two integrable functions f and g as

$$d(f,g) = \|g - f\|$$

This then defines a metric on L^1, and in fact:

Riesz-Fischer Theorem:

(L^1, d) is complete

The space of Riemann integrable functions is incomplete:

Non-Example 1:

Let r_n be an enumeration of the rational numbers in $[0,1]$ and let

$$f_n(x) = \begin{cases} 1 & \text{on } r_1, \cdots, r_n \\ 0 & \text{otherwise} \end{cases}$$

Then (f_n) is Cauchy but the limit $f = \chi_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable (but it is Lebesgue integrable)

Fact:

If $f_n \to f$ in L^1 then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

Similarly you can define L^p with $1 \leq p < \infty$ as:

Definition:

$f \in L^p$ if $\int |f(x)|^p < \infty$ and

$$\|f\|_{L_p} = \left(\int |f(x)|^p\right)^{\frac{1}{p}}$$

With a similar proof, you can show that L^p is complete.

The space L^2 is particularly noteworthy because it is a Hilbert space, that is there is an inner product

$$(f, g) = \int f(x)\overline{g(x)}dx$$

Whose norm $\|f\|_{L_2} = \sqrt{(f, f)}$ makes L^2 complete

The case $p = \infty$ is defined a bit differently:

Definition:

$f \in L^\infty$ if there is a C such that $|f(x)| \leq C$ for a.e. x

$$\|f\|_{L^\infty} = \inf\{C \text{ such that } |f(x)| \leq C \text{ for a.e. } x\}$$

Those are called the essentially bounded functions. Here L^∞ is complete as well, but with a different proof.

6. Convergence Theorems

Finally, we present convergence theorems, which are perhaps the cornerstone of Lebesgue integration theory.

Question: If $f_n \to f$ pointwise, does $\int f_n dx \to \int f dx$?

We have previously seen that the answer is **NO** in general but **YES** if $f_n \to f$ uniformly on $[a,b]$

The following guarantee $\int f_n \to \int f$ but with milder conditions:

Bounded Convergence Theorem:

Let $f_n : X \to \mathbb{R}$ measurable and suppose there is $C > 0$ such that for all n and x we have

$$|f_n(x)| \leq C$$

If $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof-Sketch: Use Egorov's Theorem and that $f_n \to f$ uniformly implies $\int f_n \to \int f$

Fatou's Lemma:

Let $f_n : X \to \mathbb{R}$ be measurable with $f_n \geq 0$. Then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Proof: By **STEP 3** of the construction, for any $g \leq \liminf_{n\to\infty} f_n$ with bounded support, let $g_n =: \min(g, f_n)$, then $g_n \to g$ a.e. so by BCT

$$\int_X g_n \to \int_X g$$

By construction $g_n \leq f_n$ and so $\int_X g_n \leq \int_X f_n$ and so taking $\liminf$ we get

$$\liminf_{n\to\infty} \int_X g_n \leq \liminf_{n\to\infty} \int_X f_n$$

$$\int_X g = \lim_{n\to\infty} \int_X g_n = \liminf_{n\to\infty} \int_X g_n \leq \liminf_{n\to\infty} \int_X f_n$$

$$\int_X g \leq \liminf_{n\to\infty} \int_X f_n$$

Taking the sup over g yields the result □

Application: This is **INCREDIBLY** useful in the calculus of variations and PDE, which deals with minimizing integrals. Usually, the best you can do is to find sequence f_n of minimizers that converges to some f. Fatou says that $\int f$ is even smaller than all the $\int f_n$ (in the $\liminf$ sense) and so f is usually the minimizer you're looking for!

Monotone Convergence Theorem:

Let $f_n : X \to \mathbb{R}$ be measurable with $f_n \geq 0$. If $f_n \nearrow f$ pointwise, then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

where this can be infinite.

Proof: Since $f_n(x) \leq f(x)$ a.e. we have $\int f_n \leq \int f$ and taking $\limsup$:

$$\limsup_{n\to\infty} \int_X f_n \leq \int_X f$$

But then by Fatou we have

$$\int_X f \leq \liminf_{n\to\infty} \int_X f_n \leq \limsup_{n\to\infty} \int_X f_n \leq \int_X f \quad \square$$

Finally, the Dominated Convergence Theorem is a generalization of the Bounded convergence theorem, where the constant C is replaced by any integrable function $g(x)$

Dominated Convergence Theorem:

Let $f_n : X \to \mathbb{R}$ be measurable with $f_n(x) \to f(x)$ pointwise

If there exists an integrable function g such that $|f_n| \leq g$ for all n, then f is integrable and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof-Sketch: You start by truncating g and then use absolute continuity of g and the Bounded Convergence Theorem

This is *the* quintessential theorem that allows us to interchange limits and integrals, and used all over again in Analysis and PDE.

Here is a simple application:

Example:

Here let $X = \mathbb{R}$ and let f be C^1 with bounded derivative

Show that, given any integrable function $p(x)$ we have

$$\lim_{h\to 0} \int_{\mathbb{R}} \left(\frac{f(x+h) - f(x)}{h} \right) p(x) dx = \int_{\mathbb{R}} f'(x) p(x) dx$$

First of all, notice

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)p(x) = f'(x)p(x) \text{ pointwise}$$

By the Mean-Value-Theorem, we have

$$\frac{f(x+h)-f(x)}{h} = f'(c) \text{ for some } c$$

Hence $$\left|\left(\frac{f(x+h)-f(x)}{h}\right)p(x)\right| = |f'(c)|\,|p(x)| \leq C\,|p(x)| \;\; (f' \text{ is bounded})$$

Since $C\,|p(x)|$ is integrable, the result follows from a continuous analog of the Dominated Convergence Theorem

The End