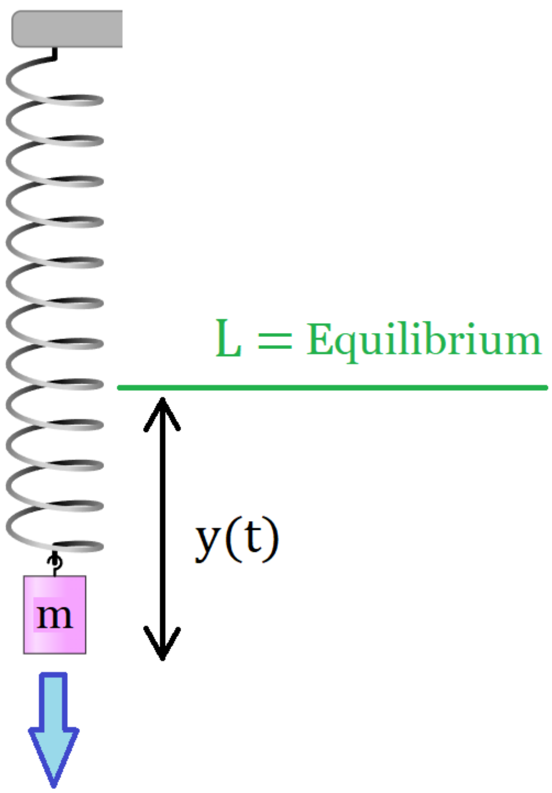
LECTURE: MECH AND FORCED VIBRATIONS

Today: Cool Application of Second-Order ODE

1. Masses and Springs

Demo: Masses and Springs (click on "Intro" and on "Lab")

Set-up: Suppose you have a mass m attached to a spring.



At first glance, the motion looks sinusoidal

Let $y(t) =$ displacement of the spring from its equilibrium position L

$$y(t) = s(t) - L$$

Where $s(t)$ is position and L is the equilibrium

Then y satisfies the following second-order ODE:

Mass Spring Equation:

$$m\, y'' + \gamma\, y' + k\, y = F(t)$$

- $m =$ mass
- $\gamma =$ damping constant, think friction
- $k =$ spring constant, depends only on the spring material
- $F(t) =$ external force, think pushing on a swing

2. Derivation

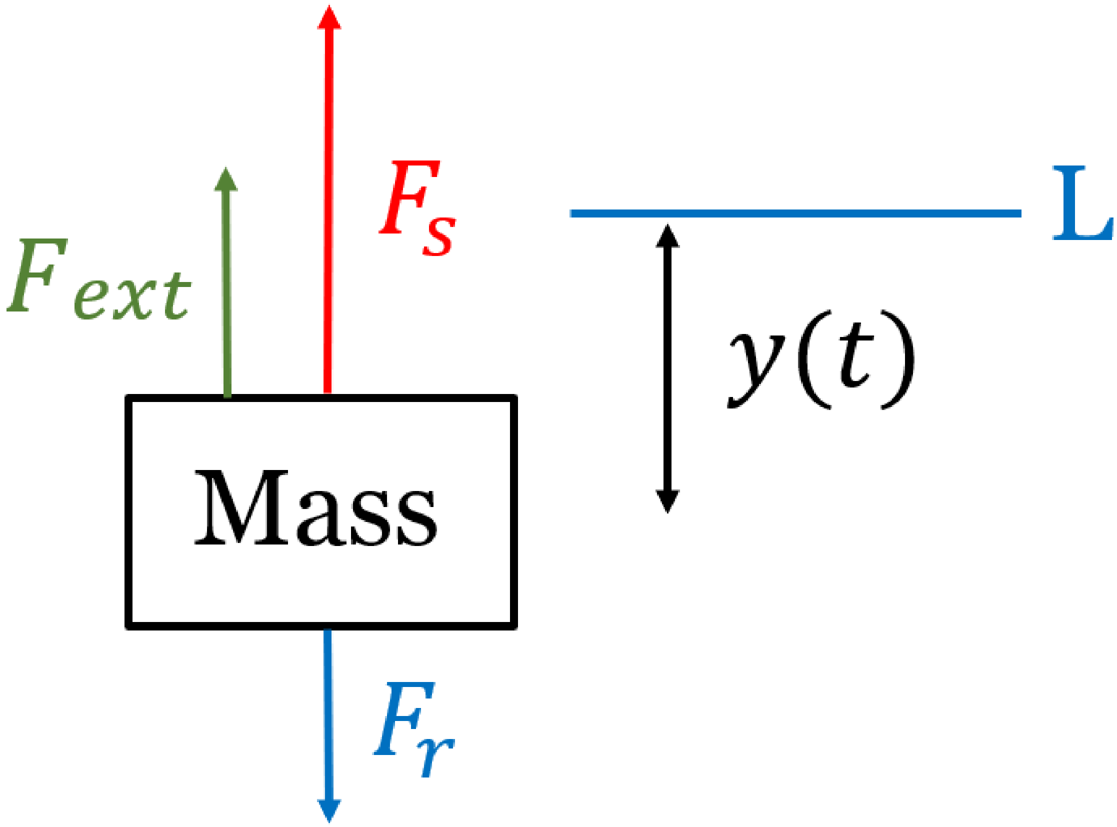
STEP 1: Start with Newton's second law of motion

$$F = ma$$

Since the displacement is $s(t) = y(t) + L$ we have

$$a(t) = s''(t) = (y(t) + L)'' = y''(t)$$

STEP 2: There are three forces acting on the spring:



(1) String Force: The force F_s pulling the string up

This is given by **Hooke's Law**

$$F_s = -ky$$

where $k > 0$ is a constant depending on the material of the spring.

This makes sense, the more you pull the string down, the more it has a tendency to bounce back up.

(2) Damping/Friction: The air resistance force F_r

$$F_r = -\gamma y'$$

Where $\gamma > 0$ is a positive constant. This is because the faster the string moves, the more friction there is

(3) External Force $F_{\text{ext}} = F(t)$ (given)

Note: We're not taking into account the effect of gravity here. It's not like the ball example that falls down because of gravity. Here if we don't touch the spring it just stays at the equilibrium position.

STEP 3: Putting everything together, we get

$$\begin{aligned} ma =& F \\ my'' =& F_s + F_r + F_{\text{ext}} \\ my'' =& -ky - \gamma y' + F(t) \\ my'' + ky + \gamma y' =& F(t) \end{aligned}$$

3. Free Vibrations

We can then predict the different kinds of scenarios that can happen:

Example 1:

$$\begin{cases} 2y'' + 18y = 0 \\ y(0) = 2 \\ y'(0) = 6\sqrt{3} \end{cases}$$

This is undamped $\gamma = 0$ spring motion without forcing $F(t) = 0$

STEP 1: Auxiliary

$$2r^2 + 18 = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = \pm 3i$$

$$y = A\cos(3t) + B\sin(3t)$$

STEP 2: Initial Condition:

$$y(0) = 2 \Rightarrow A\cos(0) + B\sin(0) = 2 \Rightarrow A = 2$$

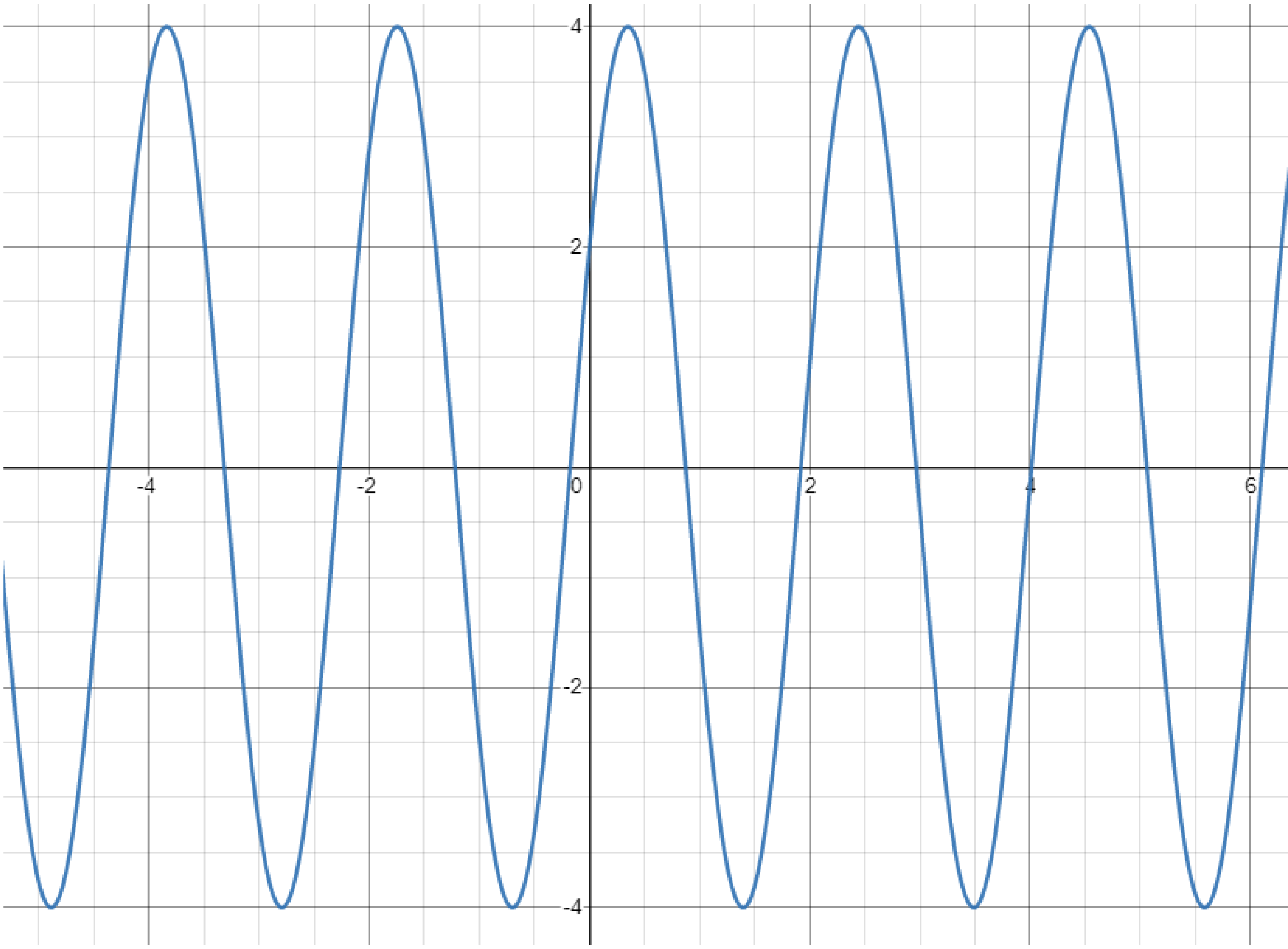
$$y = 2\cos(3t) + B\sin(3t)$$

$$y' = -6\sin(3t) + 3B\cos(3t)$$

$$y'(0) = -6(0) + 3B(1) = 3B = 6\sqrt{3} \Rightarrow B = 2\sqrt{3}$$

STEP 3: Solution

$$y = 2\cos(3t) + 2\sqrt{3}\sin(3t)$$



Notice in fact that this looks like a cosine wave

Some optional lingo:

- The **frequency** is $\omega = 3$, the 3 in $\cos(3t)$
- The **period** is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$

- The **amplitude** is

$$R = \sqrt{A^2 + B^2} = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4 + 12} = 4$$

- The **phase shift** is

$$\delta = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

- From precalculus, one may write the solution as

$$y = R\cos(\omega t - \delta) = 4\cos\left(3t - \frac{\pi}{3}\right)$$

Moral: If there is no damping and no external forcing, the motion is sinusoidal, like a cos wave

4. Damping

What if now have we damping due to friction, that is $\gamma > 0$?

In that case, the motion is sinusoidal, but gets damped

Example 2:

$$\begin{cases} 9y'' + 6y' + 37y = 0 \\ y(0) = 4 \\ y'(0) = -2 \end{cases}$$

STEP 1: Auxiliary Equation

$$9r^2 + 6r + 37 = 0$$

$$
\begin{aligned}
r =& \frac{-6 \pm \sqrt{6^2 - 4(9)(37)}}{2(9)} \\
=& \frac{-6 \pm \sqrt{-1296}}{18} \qquad \left(-1296 = -36^2\right) \\
=& \frac{-6 \pm 36i}{18} \\
=& -\frac{1}{3} \pm 2i
\end{aligned}
$$

$$y = Ae^{-\frac{t}{3}} \cos(2t) + Be^{-\frac{t}{3}} \sin(2t)$$

$$y(0) = Ae^0 \cos(0) + Be^0 \sin(0) = A = 4 \Rightarrow A = 4$$

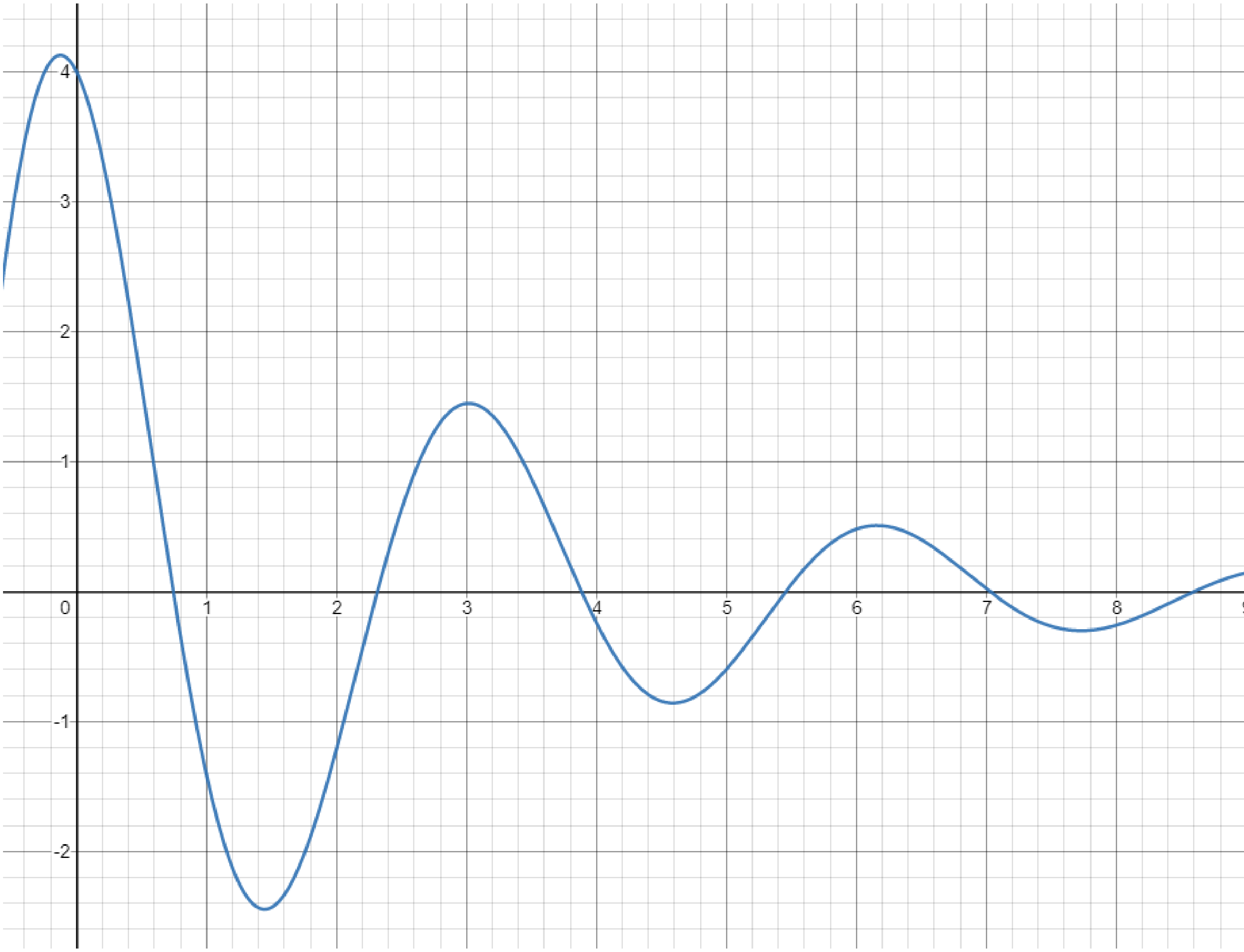
$$y = 4e^{-\frac{t}{3}} \cos(2t) + Be^{-\frac{t}{3}} \sin(2t)$$

$$y' = -\frac{4}{3}e^{-\frac{t}{3}} \cos(2t) - 8e^{-\frac{t}{3}} \sin(2t) - \frac{B}{3}e^{-\frac{t}{3}} \sin(2t) + 2Be^{-\frac{t}{3}} \cos(2t)$$

$$y'(0) = -\frac{4}{3}e^0 \cos(0) - 8e^0 \sin(0) - \frac{B}{3}e^0 \sin(0) + 2Be^0 \cos(0) = -\frac{4}{3} + 2B = -2$$

$$2B = -2 + \frac{4}{3} = -\frac{2}{3} \Rightarrow B = -\frac{1}{3}$$

$$y = 4e^{-\frac{t}{3}} \cos(2t) - \frac{1}{3}e^{-\frac{t}{3}} \sin(2t)$$



This explains *precisely* what is happening here: The motion is sinusoidal and then damps off until it becomes constant (in the limit)

5. Forcing

What is now we're adding a forcing term $F(t)$? Think swinging back and forth on a swing, using your feet to propel it

Example 3:

$$\begin{cases} y''+4y = 5\cos(3t) \\ y(0) = 1 \\ y'(0) = -4 \end{cases}$$

STEP 1: Homogeneous Solution

$$\textbf{Aux: } r^2 + 4 = 0 \Rightarrow r = \pm 2i$$

$$y_0(t) = A\cos(2t) + B\sin(2t)$$

STEP 2: Particular Solution

The right hand side is $5\cos(3t) \rightsquigarrow r = \pm 3i$ which does not coincide with the homogeneous root $r = \pm 2i$, so we guess

$$y_p = A\cos(3t) + B\sin(3t)$$

$$\begin{aligned} y'' + 4y &= 5\cos(3t) \\ (A\cos(3t) + B\sin(3t))'' + 4(A\cos(3t) + B\sin(3t)) &= 5\cos(3t) \\ (-9A\cos(3t) - 9B\sin(3t)) + (4A\cos(3t) + 4B\sin(3t)) &= 5\cos(3t) \\ -5A\cos(3t) - 5B\sin(3t) &= 5\cos(3t) \end{aligned}$$

$$\begin{cases} -5A = 5 \Rightarrow A = -1 \\ -5B = 0 \Rightarrow B = 0 \end{cases}$$

$$y_p(t) = -\cos(3t)$$

STEP 3: General Solution

$$y(t) = A\cos(2t) + B\sin(2t) - \cos(3t)$$

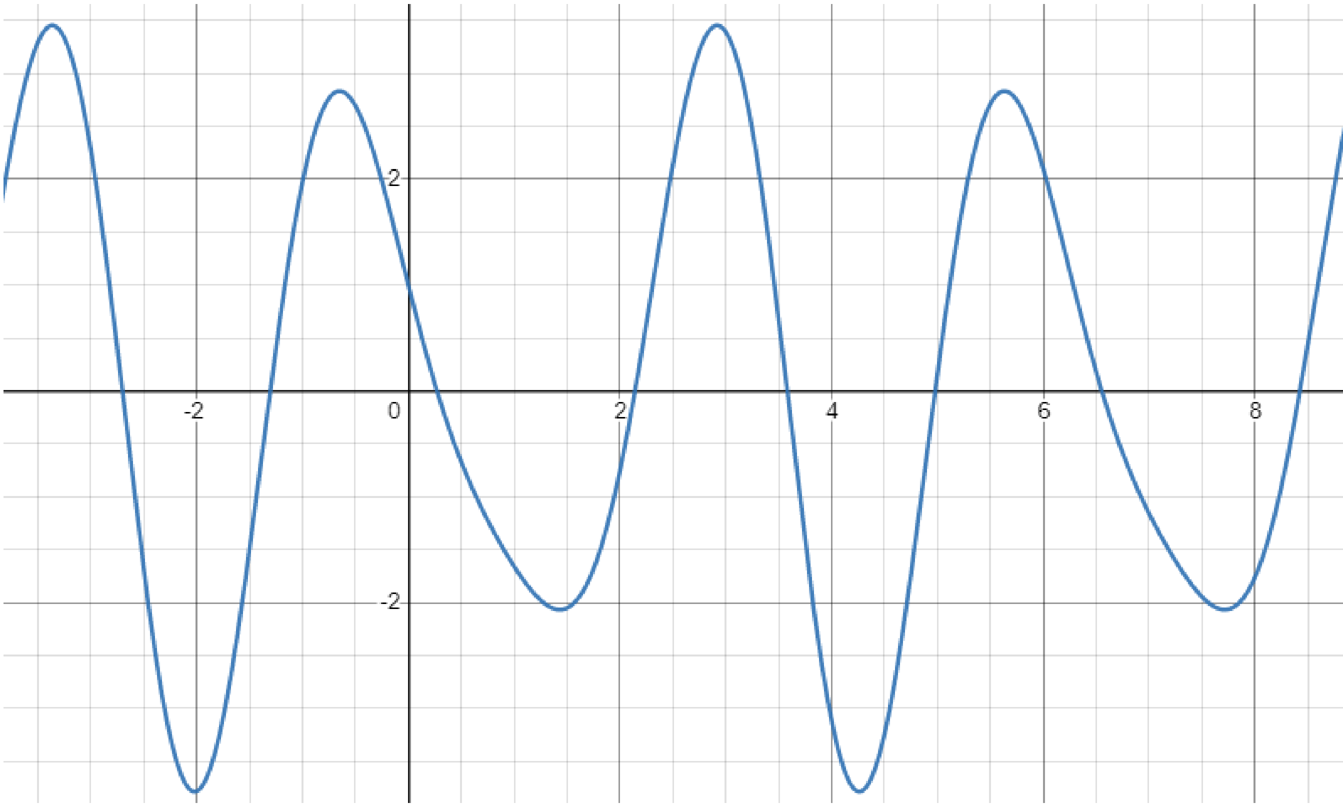
STEP 4: Initial Condition

$$\begin{aligned} y(0) =& 1 \\ A\cos(0) + B\sin(0) - \cos(0) =& 1 \\ A - 1 =& 1 \\ A = 2 \end{aligned}$$

$$y'(t) = -2A\sin(2t) + 2B\cos(2t) + 3\sin(3t)$$

$$\begin{aligned} y'(0) =& -4 \\ -2A\sin(0) + 2B\cos(0) - 3\sin(0) =& -4 \\ 2B =& -4 \\ B =& -2 \end{aligned}$$

$$y = 2\cos(2t) - 2\sin(2t) - \cos(3t)$$



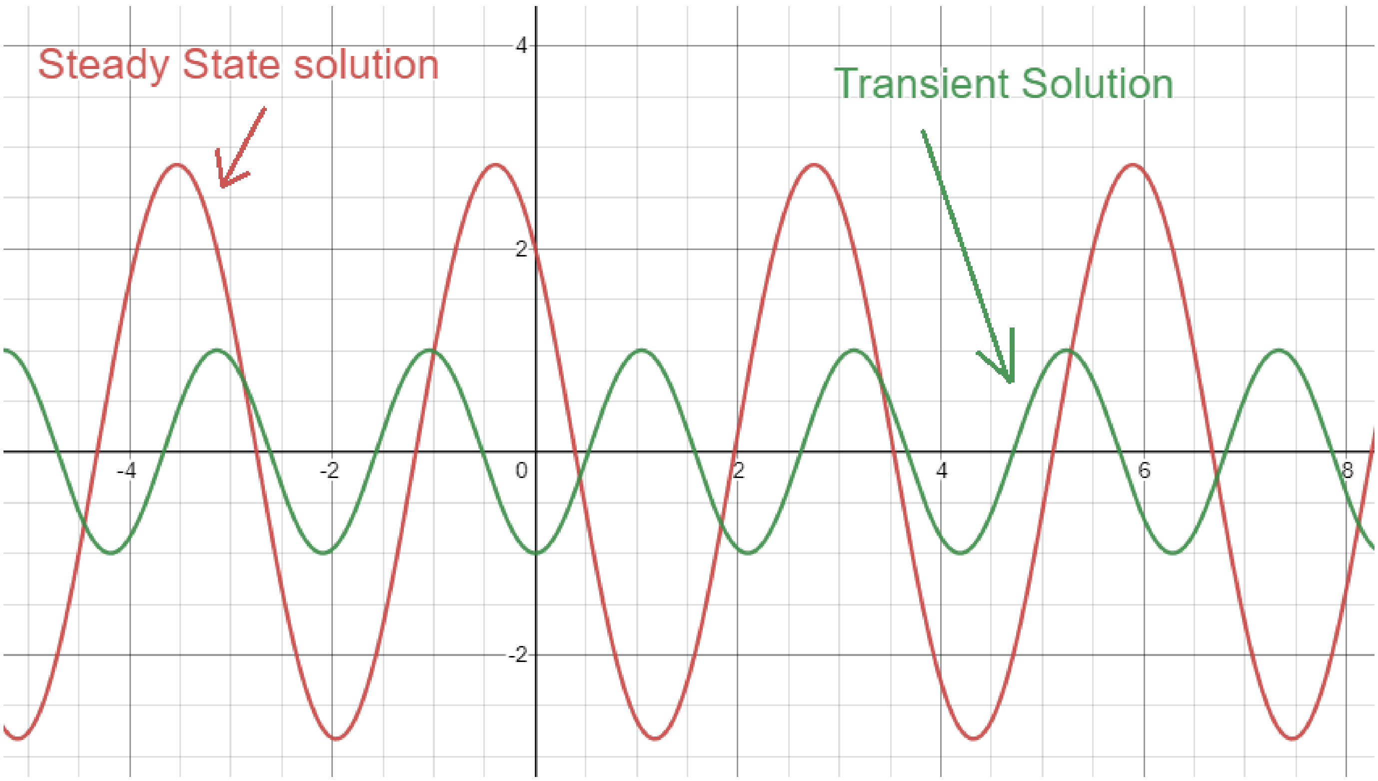
The motion still looks more or less sinusoidal. The $5\cos(3t)$ term has *some* effect on $u(t)$ but not *too* much of an effect.

Remark:

$y_0(t) = 2\cos(2t) - 2\sin(2t)$ is called the **steady-state** solution, the original solution without forcing

$y_p(t) = -\cos(3t)$ is called the **transient** solution, which is the effect due to forcing

$y = y_0 + y_p$ says that the full solution is the sum of the steady-state solution and the transient one



6. Resonance

In the previous example, the forcing frequency $r = 3i$ does not coincide with the frequency $r = 2i$ of the spring. What if they do coincide? Then we have **resonance**, and all hell breaks loose.

Example 4:

$$\begin{cases} y'' + 4y = 12\cos(2t) \\ y(0) = 1 \\ y'(0) = 6 \end{cases}$$

STEP 1: Homogeneous Solution

$$y_0(t) = A\cos(2t) + B\sin(2t)$$

STEP 2: Particular Solution

The right-hand-side $12\cos(2t) \rightsquigarrow r = \pm 2i$, which coincides with the homogeneous root $r = \pm 2i$, so there is resonance and we have to guess:

$$y_p(t) = At\cos(2t) + Bt\sin(2t)$$

$$\begin{aligned}
(y_p)' =& A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t) \\
(y_p)'' =& -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) \\
& + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t) \\
=& -4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t)
\end{aligned}$$

$$(y_p)'' + 4(y_p) = 3\cos(2t)$$

$$\begin{aligned}
(-4A\sin(2t)\cancel{-4At\cos(2t)} + 4B\cos(2t)\cancel{-4Bt\sin(2t)}) \\
+4\left(\cancel{At\cos(2t)} + \cancel{Bt\sin(2t)}\right) = 12\cos(2t)
\end{aligned}$$

$$4B\cos(2t) - 4A\sin(2t) = 12\cos(2t) + 0\sin(2t)$$

$$\begin{cases} 4B = 12 \Rightarrow B = 3 \\ -4A = 0 \Rightarrow A = 0 \end{cases}$$

$$y_p(t) = 3t\sin(2t)$$

STEP 3: General Solution

$$y = A\cos(2t) + B\sin(2t) + 3t\sin(2t)$$

STEP 4: Initial Condition

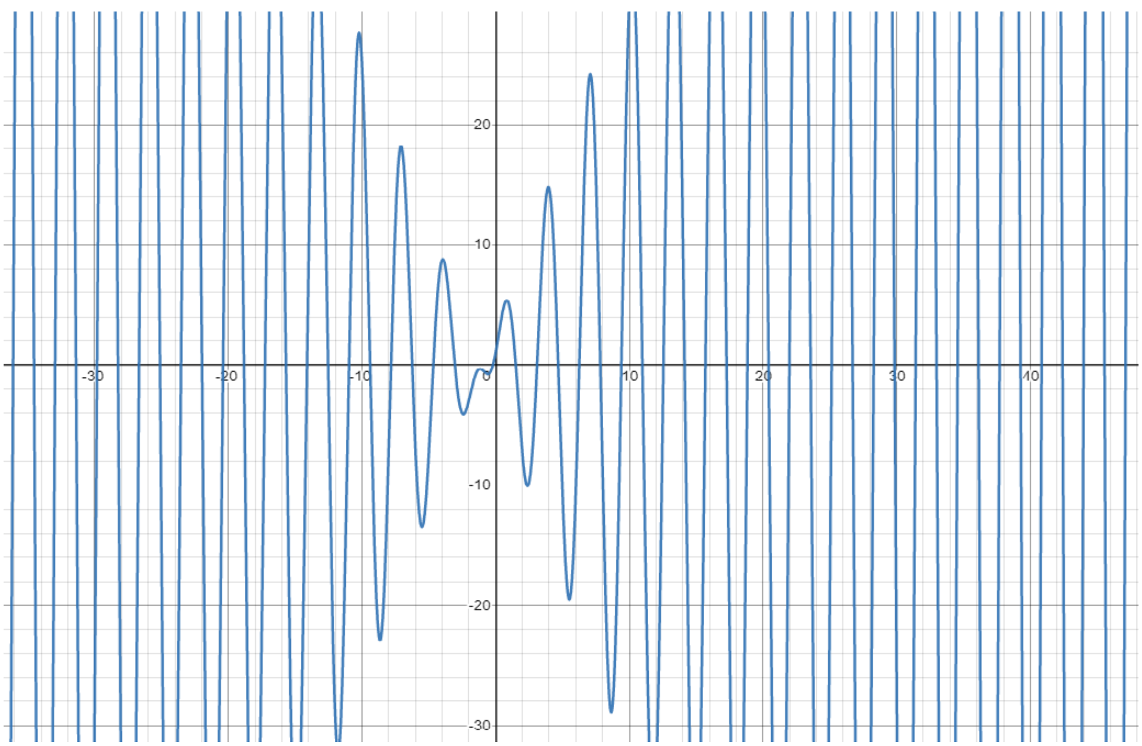
$$y(0) = 1 \Rightarrow A\cos(0) + B\sin(0) + 0\sin(0) = A = 1$$

$$y(t) = \cos(2t) + B\sin(2t) + 3t\sin(2t)$$
$$y'(t) = -2\sin(2t) + 2B\cos(2t) + 3\sin(2t) + 6t\cos(2t)$$

$$y'(0) = 6 \Rightarrow -2\sin(0) + 2B\cos(0) + 3\sin(0) + 0\cos(0) = 2B = 6 \Rightarrow B = 3$$

STEP 5: Answer:

$$y = \cos(2t) + 3\sin(2t) + 3t\sin(2t)$$



In this case there is **resonance** and the solution blows up!! This is due to the resonance term $3t\sin(2t)$ and *very* different from the previous example.

Application 1: This explains that when you swing on a swing in a playground and move your feet *just* at the right frequency, you get yourself in a larger and larger swing.

Application 2: Resonance can cause bridges to collapse! This is what happened for example in the Tacoma Narrows Bridge in 1940, where the frequency of the wind was equal to the natural frequency of the bridge, causing it to crash (fortunately no human lives were lost)

Video: Resonance